Home assignment 5: Convergence for Lipschitz maps

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

Remark 5.1. For all metric spaces in this assignment, we assume that they admit a countable dense set (that is, are "second countable").

Definition 5.1. **A path** in a metric space M is a continuous map $\gamma: [0,1] \longrightarrow M$ **Length** of a path is $\sup_{0=x_1<x_2<...<x_n=1\in[0,1]} \sum_{i=1}^{n-1} d(\gamma(x_i), \gamma(x_{i+1}))$.

Exercise 5.1. Let $\gamma: [0,1] \longrightarrow \mathbb{R}^2$ be a path of length 1, with $d(\gamma(0), \gamma(1)) > 1-\varepsilon^2$. Prove that the image of γ is contained in a rectangle of size $\varepsilon \times (1+\varepsilon)$.

Exercise 5.2. Let $\gamma: [0,1] \longrightarrow \mathbb{R}^2$ be a path of finite length. Prove that the image of γ has measure 0.

Hint. Use the previous exercise.

Definition 5.2. Let M be a metric space, and $\operatorname{Map}(X, M)$ the set of all continuous maps from a set X to M. Define the metric on $\operatorname{Map}(X, M)$ using $d(f_1, f_2) = \sup_{x\in X} d(f_1(x), f_2(x))$. The corresponding topology is called **the uniform topology** on $\operatorname{Map}(X, M)$.

Exercise 5.3. Prove that the length of a path is continuous in the uniform topology on $\operatorname{Map}[([0,1], M)$ on the space $\operatorname{Map}_C([0,1], M)$ of C-Lipschitz paths, or find a counterexample.

Exercise 5.4. Let M be a manifold with a marked point m, and $\operatorname{Map}((S^1, 0), (M, m))$ the space of maps putting $0 \in S^1$ to $m \in M$, where M is a manifold. We equip $\operatorname{Map}((S^1, 0), (M, m))$ with the uniform topology. Denote by W the set of connected components of $\operatorname{Map}((S^1, 0), (M, m))$. Construct a bijective equivalence between W and the fundamental group $\pi_1(M, m)$.

Definition 5.3. Let M be a topological space, and $U \subset M$ an open subset. Given $x \in X$, we define a subset $S_{x,U} \subset \operatorname{Map}(X, M)$ as the set of all maps putting x to U. **Tychonoff topology**, also called **topology of pointwise convergence**, is topology where open subsets are obtained by finite intersections and all unions of $S_{x,U}$ for all possible x and U.

Exercise 5.5. Let $\{f_i\} \in \operatorname{Map}(X, M)$ be a sequence of maps, with M being a topological space. Prove that $\{f_i\}$ converges to f in Tychonoff topology if and only if for each $x \in X$ one has $\lim_i f_i(x) = f(x)$.

Exercise 5.6. Construct a sequence of functions $\{f_i \in C^\infty([0,1])\}$ which satisfy $\int_0^1 |f_i(t)| dt = 1$ and converge pointwise to 0.