Home assignment 6: Isometries of $\mathbb{H}^2$.

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

Definition 6.1. **Order** of $A \in GL(n)$ is the smallest positive integer such that $A^k = \mathsf{Id}$.

Exercise 6.1. Let A be an element of finite order k in $GL(2, \mathbb{Z})$. Prove that $k = 2, 3, 4, 6$.

Exercise 6.2. Let $A \subset SL(3, \mathbb{Z})$ be an element of finite order k in $SL(3, \mathbb{Z})$. Prove that $k = 2, 3, 4, 6$.

Exercise 6.3. Find an element of order 5 in $GL(4, \mathbb{Z})$, or prove that it does not exist.

Remark 6.1. Let $V = \mathbb{R}^3$ be a vector space with quadratic form q of signature (1,2), $V^+ := \{v \in V \mid q(v) > 0\}$, and $\mathbb{P}V^+$ its projectivisation. Then $\mathbb{P}V^+ = SO^+(1,2)/SO(1)$, giving $\mathbb{P}V^+ = \mathbb{H}^2$; this is one of the standard models of a hyperbolic plane. **The absolute** is projectivization of the set of all isotropic lines; it is identified with the boundary of $\mathbb{P}V^+$ in $\mathbb{P}V$.

Definition 6.2. Let $l \subset V$ be a line, that is, a 1-dimensional subspace. The property $q(x,x) < 0$ for a non-zero $x \in l$ is written as $q(l,l) < 0$. A line l with $q(l,l) < 0$ is called **negative line**, a line with $q(l,l) > 0$ is called **positive line**.

Remark 6.2. Negative lines bijectively correspond to geodesics in $\mathbb{P}V^+ = \mathbb{H}^2$ (Lecture 8): an orthogonal complement to a negative line is a 2-dimensional plane $l^\perp$, its projectivization intersected with $\mathbb{P}V^+ = \mathbb{H}^2$ is a geodesic.

Exercise 6.4. Let γ_1, γ_2 be geodesics on a hyperbolic plane, and l_1, l_2 the corresponding negative lines.

a. Prove that l_1 is orthogonal to l_2 if and only if γ_1 is orthogonal to γ_2.

b. Prove γ_1 intersects γ_2 if and only if the 2-plane $\langle l_1, l_2 \rangle$ generated by l_1, l_2 has signature (0,2).

c. Prove that γ_1 and γ_2 passes through the same point on the absolute if and only if the 2-plane generated by l_1, l_2 has degenerate scalar product.

d. Prove that the angle between γ_1 and γ_2 divides $\frac{2\pi}{k}$ if and only if the angle between l_1, l_2 in $\langle l_1, l_2 \rangle$ divides $\frac{2\pi}{k}$.

e. Prove that the angle between γ_1 and γ_2 is equal to the angle between l_1, l_2 in $\langle l_1, l_2 \rangle$.

Remark 6.3. Recall that $h \in SO^+(1,2)$ is called **hyperbolic** if it has an eigenvalue α with $|\alpha| > 1$.

Exercise 6.5. Let q be a quadratic form of signature (1,2) on $\mathbb{R}^3$ with integer coefficients, $h \in SO^+(1,2)$ a hyperbolic matrix with integer coefficients, and $P_h(t)$ its characteristic polynomial. Prove that $P_h(t)$ has precisely 1 rational root.