Home assignment 7: Coverings.

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

Definition 7.1. **A covering** is a continuous map $\pi : M_1 \longrightarrow M$ such that for each $x \in M_1$ there exists a neighbourhood $U \subset M$ such that $\pi : \pi^{-1}(U) = U \times S$, where S is a fixed set with discrete topology, and the map $\pi : \pi^{-1}(U) = U \times S \longrightarrow U$ is a projection. **A morphism of coverings** $\pi_1 : M_1 \longrightarrow M$, $\pi_2 : M_2 \longrightarrow M$ is a map $f : M_1 \longrightarrow M_2$ which commutes with the projections to M. A covering $M_1 \longrightarrow M$ is called **universal covering** if M_1, M are connected, and M_1 is simply connected.

Definition 7.2. A continuous map $\pi : M_1 \longrightarrow M$ is called **proper** if for any compact $K \subset M$, the preimage $\pi^{-1}(K)$ is compact.

Exercise 7.1. Let $\pi : M \longrightarrow M_1$ be a smooth map of n-manifolds with differential non-degenerate everywhere. Assume that π is proper. Prove that π is a covering.

Definition 7.3. Let $\pi_1 : M_1 \longrightarrow M$, $\pi_2 : M_2 \longrightarrow M$ be continuous maps. **Fibered product** $M_1 \times_M M_2$ is the subset of $M_1 \times M_2$ defined as $M_1 \times_M M_2 := \{(x, y) \in M_1 \times M_2 \mid \pi_1(x) = \pi_2(y)\}$, with induced topology.

Exercise 7.2. Let $\pi_1 : M_1 \longrightarrow M$, $\pi_2 : M_2 \longrightarrow M$ be coverings. Prove that $M_1 \times_M M_2 \longrightarrow M$ is also a covering.

Exercise 7.3. Let $\pi_1 : M_1 \longrightarrow M$, $\pi_2 : M_2 \longrightarrow M$ be universal coverings. Prove that $M_1 \times_M M_2 \longrightarrow M$ is a union of $\pi_1(M)$ disconnected copies of M_1. Deduce that the universal covering is unique up to an isomorphism of coverings.

Exercise 7.4. Let $\pi : M_1 \longrightarrow M$ be the universal covering. Prove that the group $\operatorname{Aut}_M(M_1)$ of its automorphisms (in the category of coverings) is isomorphic to $\pi_1(M)$.

Exercise 7.5. Let $\pi : M_1 \longrightarrow M$ be a connected covering, and $u : M_u \longrightarrow M$ be the universal covering. Prove that $M_u \times_M M_1$ is a disconnected sum of several copies of M_u. Prove that the map $u : M_u \longrightarrow M$ can be factorized through $\pi : M_1 \longrightarrow M$, with u equal to a composition $M_u \xrightarrow{\phi} M_1 \xrightarrow{\pi_1} M$.

Exercise 7.6. Let $\pi : M_1 \longrightarrow M$ be a connected covering. Prove that π induces a group monomorphism $\pi_1(M_1) \longrightarrow \pi_1(M)$. Prove that the isomorphism classes of connected coverings $\pi : M_1 \longrightarrow M$ are in bijective correspondence with subgroups of $\pi_1(M)$.

Exercise 7.7. Let $x \in M$ be a point in a connected manifold M. Consider a functor Φ from the category $\mathcal{C}_M$ of coverings of M to the category of sets mapping a covering $\pi : M_1 \longrightarrow M$ to $\pi^{-1}(x)$. Prove that the fundamental group $\pi_1(M, x)$ naturally acts on the set $\pi^{-1}(x)$. Prove that Φ defines an equivalence of categories from $\mathcal{C}_M$ to the category of sets with an action of the group $\pi^{-1}(x)$.