# Home assignment 8: Foliations.

**Rules:** This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

**Definition 8.1.** Let $B \subset TM$ be an involutive sub-bundle. The corresponding **(smooth) foliation** is a collection of immersed submanifolds $X \stackrel{\phi}{\hookrightarrow} M$ which are tangent to $B$ everywhere, that is, $\phi(T_xX) = B\big|_x$. These submanifolds are called **leaves** of the foliation, and $B$ its **tangent bundle**. Its **rank** is $\operatorname{rk} B$. Frobenius theorem claims that $M$ is the union of all leaves of $\mathcal{F}$. The **leaf space** is the set of all leaves of $\mathcal{F}$ with the quotient topology (a set is open if its preimage in $M$ is open).

**Exercise 8.1.** Construct a foliation on a simply connected, compact manifold with all leaves non-compact.

**Exercise 8.2.** Prove that $\mathbb{C}P^n$ does not admit a rank 1 foliation.

**Exercise 8.3.** Let $\mathcal{F}$ be a foliation with compact leaves. Assume that a compact group $G$ acts on $M$ preserving the fibers and transitively on each fiber. Prove that its leaf space is Hausdorff.

**Exercise 8.4.** Construct a rank 1 foliation with compact leaves on a 3-sphere $M$, such that the projection $M \longrightarrow M/\mathcal{F}$ to its leaf space is not a smooth submersion.

**Exercise 8.5.** Find a foliation on a compact manifold with all leaves compact, and not all of them diffeomorphic.