Home assignment 9: Connections and curvature.

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

Exercise 9.1. Let (B, ∇) be a rank 1 bundle with connection over a circle S^1, and $\nabla' := \nabla + \theta$, where $\theta \in \Lambda^1(S^1)$ is a 1-form, understood as a section of $\Lambda^1(S^1) \otimes \operatorname{End}(B)$. Prove that holonomy of ∇' is equal to holonomy of ∇ if and only if θ is exact.

Exercise 9.2. Let (B, ∇) be a bundle with connection over M, and $\nabla' := \nabla + \theta$, where $\theta \in \Lambda^1(M)$ is a 1-form, understood as a section of $\Lambda^1(S^1) \otimes \mathsf{Id}_B$, where $\mathsf{Id}_B \in \operatorname{End}(B)$ is the identity section. Assume that both ∇ and ∇' are flat. Prove that $d\theta = 0$.

Exercise 9.3. Let B be a trivial real vector bundle over a circle S^1, $x \in S^1$ a point, and $A \in \operatorname{Aut}\left(B\big|_x\right)$ a linear automorphism with all eigenvalues real and positive, or all eigenvalues complex. Construct a connection ∇ such that its holonomy in x is equal to A.

Exercise 9.4. Let B be a rank 1 bundle on $\mathbb{R}^2$, and $V \in \Lambda^2(\mathbb{R}^2)$ a 2-form. Construct a connection on B such that its curvature is equal to V.

Exercise 9.5. Let B be a bundle on $\mathbb{R}^2$, and $R \in \Lambda^2(T_0^*\mathbb{R}^n) \times \operatorname{End}\left(B\big|_0\right)$ a 2-form on the tangent vector space in 0 with values in $\operatorname{End}\left(B\big|_0\right)$. Construct a connection with curvature Θ such that $\Theta\big|_0 = R$.

a. When $n = 2$, and R takes values in traceless endomorphisms.

b. When $n = 2$, and R arbitrary.

c. For arbitrary n and R.