Complex manifolds in dimension 1: end-term exam

Rules: Every student receives from me a list of 10 exercises (chosen randomly), and has to solve as many of them as you can before September 2024. Please write down the solution and bring it to exam for me to see. To pass the exam you are required to explain the solutions, using your notes. Please learn proofs of all results you will be using on the way (you may put them in your notes).

The final score N is obtained by summing up the points from the exam problems and the class tests, using the formula $N = 10e + t/2$, where t is the sum of class tests, e the points for exam problems. Marks: C when $30 \leqslant N < 40$, B when $40 \leqslant N < 60$, A when $60 \leqslant N \leqslant 80$, A+ when $N > 80$.

1 Almost complex structures and holomorphic functions

Exercise 1.1. Let f be a smooth real function on a disk D such that $dId(f)$ is a nowhere degenerate 2-form of positive orientation. Prove that f cannot have a maximum anywhere on D.

Definition 1.1. A function f on an almost complex manifold (M, I) is **holomorphic** if $df \in \Lambda^{1,0}(M, I)$.

Exercise 1.2. Let M be a simply connected almost complex manifold, and η a non-zero closed (1,0)-form. Prove that M admits a non-constant holomorphic map to $\mathbb{C}^* = \mathbb{C}\backslash 0$.

Exercise 1.3. Let M be an almost complex manifold, and $\phi : M \longrightarrow \mathbb{R}$ a non-constant function which satisfies $dId(\phi) = 0$. Prove that M admits a non-zero holomorphic 1-form.

Exercise 1.4 (2 points). Let $\Gamma \subset \mathrm{Aut}(\Delta)$ be a discrete, cyclic subgroup in the group of automorphisms of a disk. Prove that Δ admits a Γ-invariant holomorphic 1-form, or find a counterexample.

Exercise 1.5 (2 points). Let M_0 be a Riemann surface equipped with a holomorphic action of a group $\Gamma = \mathbb{Z}$, generated by an automorphism γ. Consider a nowhere vanishing function ϕ on M_0 such that $\gamma^*(\phi) = \mathsf{const} \cdot \phi$ and $dId\phi = 0$. Prove that M admits a Γ-invariant holomorphic 1-form.

Exercise 1.6. Let M be a Riemannian 2-manifold, and $f : M \longrightarrow M$ a conformal map preserving the Riemannian volume. Prove that f is an isometry.

Exercise 1.7. Let η be a closed 1-form on a 1-dimensional complex manifold (M, I) such that $I(\eta)$ is also closed.

a. Prove that $\eta = df$, where f is a real part of a holomorphic function if M is simply connected.

b. Find an example of (M, I, η) when η is exact, but such f does not exist.

2 Holomorphic functions on complex manifolds

Exercise 2.1 (2 points). Let f be a real-valued smooth function on a complex manifold which satisfies $dId(f) = 0$. Prove that f is a real part of a holomorphic function or find a counterexample.

Exercise 2.2. Let f be a holomorphic function on a disk Δ. Prove that $\int_\Delta f\omega = \pi f(0)$, where $\omega = dx \wedge dy$ is the standard volume form.

Exercise 2.3. Let M be a simply connected complex manifold, and θ a non-zero exact 1-form such that $d(I\theta) = 0$. Prove that M admits a non-constant holomorphic function.

Exercise 2.4. Let f be a holomorphic function on a Riemann surface such that $|f|$ and df is nowhere zero. Prove that $|f|$ is smooth. Prove that $dId(|f|)$ is a nowhere degenerate 2-form.

Exercise 2.5. Let f be a non-constant holomorphic function on a disk, continuous on the boundary. Prove that $\sqrt{-1}\int_{\partial\Delta} \bar{f}\frac{df}{dz}dz > 0$.

Exercise 2.6. Let f be a holomorphic function on $\mathbb{C}$ such that $|f|(z) < |P(z)|$ for some polynomial $P(z)$. Prove that f is polynomial.

Exercise 2.7. Let f_i be a collection of holomorphic functions on a disk such that $\sum_{i=0}^\infty |f_i(z)|$ converges uniformly on Δ. Prove that $\sum_{i=0}^\infty |f_i'(z)|$ converges uniformly on Δ.

Exercise 2.8 (2 points). Construct a non-zero bounded holomorphic function on $\{z \in \mathbb{C} \ \ | \ \ \operatorname{Re}(z) > 0\}$. such that $f(n) = 0$ for all $n \in \mathbb{Z}^{>0}$.

3 Homogeneous spaces

Exercise 3.1. Consider the space $\operatorname{AdS}(2) := SO^+(1,2)/SO^+(1,1)$ (so-called "2-dimensional anti-de Sitter space"). Prove that $\operatorname{AdS}(2)$ does not admit an $SO^+(1,2)$-invariant Riemannian structure.

Exercise 3.2. Prove that $\operatorname{AdS}(2) = SO^+(1,2)/SO^+(1,1)$ is not simply connected.

Exercise 3.3. Let V be an n-dimensional Hermitian complex space of signature $(1, n-1)$. Prove that the space $B \subset \mathbb{P}_{\mathbb{C}}V$ of all positive complex lines in V is biholomorphic to a ball in $\mathbb{C}^{n-1}$.

Exercise 3.4. Let V be an odd-dimensional vector space equipped with a positive definite quadratic form. Prove that the center of $SO(V)$ is trivial.

Exercise 3.5 (2 points). Let $V = \mathbb{R}^n$ be a Euclidean vector space, and $\mathrm{Gr}(2, V) = \frac{SO(n)}{SO(2)\times SO(n-2)}$ the Grassmannian of 2-planes in V. Prove that $\mathrm{Gr}(2, V)$ admits an $SO(n)$-invariant almost complex structure. Prove that it is integrable.

Exercise 3.6. Let X be the disk Δ with Poincare metric, and S^1X the space of all vectors of length 1 in $T\Delta$. Prove that the action of $SO(1, 2)$ on S^1X is free and transitive.

Definition 3.1. **Horocycle** on a Poincaré plane is an orbit of a parabolic subgroup $P_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in PSL(2, \mathbb{R}) = SO^+(1, 2)$.

Exercise 3.7. Prove that the group of isometries $\mathrm{Iso}(\mathbb{H}^2) = SO^+(1, 2)$ acts transitively on the set of all horocycles.

Exercise 3.8. Let $V = \mathbb{R}^3$ be a vector space with quadratic form q of signature (1,2), $p_1, p_2 \in V$ two vectors with positive square, and $\tilde{p}_1, \tilde{p}_2$ the points in $\mathbb{P}V^+ = \mathbb{H}^2$ corresponding to p_1, p_2. Define **tance** of p_1, p_2 as $\mathsf{ta}(p_1, p_2) := \frac{q(p_1,p_2)^2}{q(p_1,p_1)q(p_2,p_2)}$. Prove that the distance between points $\tilde{p}_1, \tilde{p}_2$ in hyperbolic metric is a function of $\mathsf{ta}(p_1, p_2)$.

4 Poincaré metric and Kobayashi pseudometric

Exercise 4.1. Let M be a compact metric space, and $\mathrm{Iso}(M)$ the group of its isometries. We equip $\mathrm{Iso}(M)$ with topology of uniform convergence. Prove that $\mathrm{Iso}(M)$ is compact.

Exercise 4.2. Let M be a compact, Kobayashi-hyperbolic complex manifold. Prove that M does not admit non-zero holomorphic vector fields.

Definition 4.1. **A circle** with center x and radius r on a Poincaré plane $\mathbb{H}^2$ is the set $\{y \in \mathbb{H}^2 \ \mid\ d(x, y) = r\}$

Exercise 4.3. Let S be a circle on a disk $\Delta \subset \mathbb{C}$ with Poincaré metric. Prove that S is a circle in Euclidean geometry on $\Delta \subset \mathbb{C}$.

Exercise 4.4. Let r be a maximal radius of a circle in Poincaré plane which can be inscribed in a triangle. Prove that $r < \infty$.

Exercise 4.5. Let γ_1, γ_2 be two geodesics in Poincaré plane with the same end in the absolute. Prove that for each $\varepsilon > 0$ there exists $x \in \gamma_1, y \in \gamma_2$ such that $d(x, y) < \varepsilon$.

Exercise 4.6. Let $a, b, c \in \mathsf{Abs}$ be three distinct points on the absolute, and $A \in SO^+(1, 2)$ an isometry of Poincaré plane which fixes a, b, c. Prove that $A = \mathsf{Id}$.

Exercise 4.7 (2 points). Let $I_1, I_2 \in SL(2, \mathbb{R})$ be two operators satisfying $I_1^2 = I_2^2 = -\mathsf{Id}$. Prove that $|\operatorname{Tr} I_1 I_2| \geqslant 2$, with equality if and only if $I_1 = \pm I_2$.

5 Coverings, fundamental group, topology

Exercise 5.1. Let $V = \mathbb{R}^3$ be a vector space with quadratic form of signature (1,2), and $\mathbb{P}V^- \subset \mathbb{P}V = \mathbb{R}P^3$ the space of negative lines. Prove that $\mathbb{P}V^-$ is homeomorphic to a Möbius strip.

Exercise 5.2. Let M be a Riemann surface with infinite fundamental group. Prove that any continuous map $S^2 \longrightarrow M$ is homotopic to a trivial map (map to a point).

Exercise 5.3. Let M be a manifold with infinite fundamental group, $\tilde{M}$ its universal covering, and $\tilde{M} \times_M \tilde{M}$ the fibered product. Prove that $\tilde{M} \times_M \tilde{M}$ has infinitely many connected components.

Exercise 5.4. Let $\tilde{M} \longrightarrow M$ be a connected covering, and $G = \operatorname{Aut}_M(\tilde{M})$ a group of automorphisms of the covering. Prove that the action of G on $\tilde{M}$ is free, and the quotient space $\tilde{M}/G$ is also a covering of M.

Definition 5.1. **Free group** is a fundamental group of a bouquet of circles (a collection of circles glued in one point).

Exercise 5.5. Let $M_1 = S^1 \times S^1$ be a torus and M be M_1 without a point. Prove that its fundamental group is free.

Exercise 5.6. Let M be a simply connected manifold. Prove that any real rank 1 bundle on M is trivial.

Exercise 5.7. Prove that all real vector bundles on $\mathbb{R}$ are trivial. Construct a non-trivial vector bundle on S^1 or prove it does not exist.

Exercise 5.8. Let $TS^2 \oplus C^\infty S^2$ be a direct sum of a tangent bundle TS^2 and a trivial 1-dimensional bundle. Is the bundle $TS^2 \oplus C^\infty S^2$ trivial?

Exercise 5.9. Find a rank 1 sub-bundle $B \subset TS^3$ such that the corresponding foliation has at least one non-compact leaf.