Complex structures and conformal structures

lecture 1

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Complex structure on a vector space

DEFINITION: Let V be a vector space over $\mathbb{R}$, and $I: V \longrightarrow V$ an automorphism which satisfies $I^2 = -\operatorname{Id}_V$. Such an automorphism is called **a complex structure operator** on V.

We extend the action of I on the tensor spaces $V \otimes V \otimes ... \otimes V \otimes V^* \otimes V^* \otimes ... \otimes V^*$ by multiplicativity: $I(v_1 \otimes ... \otimes w_1 \otimes ... \otimes w_n) = I(v_1) \otimes ... \otimes I(w_1) \otimes ... \otimes I(w_n)$.

Trivial observations:

1. **The eigenvalues α_i of I are $\pm\sqrt{-1}$.** Indeed, $\alpha_i^2 = -1$.

2. **V admits an I-invariant, positive definite scalar product ("metric") g.** Take any metric g_0, and let $g := g_0 + I(g_0)$.

3. **I is orthogonal for such g.**
Indeed, $g(Ix, Iy) = g_0(x, y) + g_0(Ix, Iy) = g(x, y)$.

4. **I diagonalizable over $\mathbb{C}$.** Indeed, any orthogonal matrix is diagonalizable.

5. **There are as many $\sqrt{-1}$-eigenvalues as there are $-\sqrt{-1}$-eigenvalues.**

Comples structure operator in coordinates

This implies that in an appropriate basis in $V \otimes_{\mathbb{R}} \mathbb{C}$, the complex structure operator is diagonal, as follows:

$$\left[\begin{array}{cccc|cccc}
\sqrt{-1} & & & & & & & \\
& \sqrt{-1} & & & & 0 & & \\
& & \ddots & & & & & \\
& & & \sqrt{-1} & & & & \\
\hline
& & & & -\sqrt{-1} & & & \\
& 0 & & & & -\sqrt{-1} & & \\
& & & & & & \ddots & \\
& & & & & & & -\sqrt{-1}
\end{array}\right]$$

We also obtain its normal form in a real basis:

$$\begin{bmatrix}
0 & -1 & & & & & & \\
1 & 0 & & & & & & \\
& & 0 & -1 & & & & \\
& & 1 & 0 & & & & \\
& & & & \ddots & & & \\
& & & & & \ddots & & \\
& & & & & & 0 & -1 \\
& & & & & & 1 & 0
\end{bmatrix}$$

The Hodge decomposition

DEFINITION: Let (V, I) be a space equipped with a complex structure. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

REMARK: The Hodge decomposition **uniquely determines the complex structure operator.** Moreover, **any decomposition $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ such that $V^{0,1} = \overline{V^{1,0}}$ determines the complex structure operator (prove this as an exercise).**

Complex manifolds

DEFINITION: A holomorphic function on $\mathbb{C}^n$ is a smooth function $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ such that its differential df is complex linear, that is, $df \in \Lambda^{1,0}(\mathbb{C}^n)$.

DEFINITION: A map $f : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is **holomorphic** if all its coordinate components are holomorphic.

DEFINITION: A complex manifold is a manifold equipped with an atlas with charts identified with open subsets of $\mathbb{C}^n$ and transition maps holomorphic.

Complex manifolds and almost complex manifolds

DEFINITION: An **almost complex structure** on a smooth manifold is an endomorphism $I: TM \longrightarrow TM$ of it tangent bundle which satisfies $I^2 = -\operatorname{Id}_{TM}$.

DEFINITION: **Standard almost complex structure** is $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$ on $\mathbb{C}^n$ with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$.

DEFINITION: A map $\Psi: (M,I) \longrightarrow (N,J)$ from an almost complex manifold to an almost complex manifold is called **holomorphic** if its differential commutes with the almost complex structure.

DEFINITION: A complex-valued function $f \in C^\infty M$ on an almost complex manifold is **holomorphic** if df belongs to $\Lambda^{1,0}(M)$, where

$$\Lambda^1(M) \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M)$$

is the Hodge decomposition of the cotangent bundle.

REMARK: For standard almost complex structures, **this is the same as the coordinate components of Ψ being holomorphic functions.** Indeed, **a function $f: (M,I) \longrightarrow (\mathbb{C},I)$ is holomorphic if and only if its differential df satisfies $df(Iv) = \sqrt{-1}\, df(v)$.**

Integrability of almost complex structures

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof. Step 1: The complex coordinates define the Hodge decomposition $\Lambda^1(M) \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M)$, because $\Lambda^{1,0}(M)$ **is generated by differentials of holomorphic functions,** and $\Lambda^{0,1}(M)$ is its complex conjugate.
Step 2: Conversely, the Hodge decomposition determines the ring of holomorphic functions, because $f \in C^\infty_{\mathbb{C}}(M)$ is holomorphic if and only if $df \in \Lambda^{1,0}(M)$. Now, for any two sets of holomorphic coordinate functions, the transition map is clearly holomorphic, hence the Hodge decomposition defines a class of atlaces with holomorphic transition functions. ■

THEOREM: Let (M, I) be an almost complex manifold, $\dim_{\mathbb{R}} M = 2$. **Then I is integrable.**

Proof: Later in this course. **This is one of the central theorems in these lectures.** For a while, we will assume this statement without a proof.

Riemannian manifolds

DEFINITION: Let $h \in \operatorname{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: For any $x, y \in M$, and any path $\gamma : [a,b] \longrightarrow M$ connecting x and y, consider **the length** of γ defined as $L(\gamma) = \int_\gamma |\frac{d\gamma}{dt}| dt$, where $|\frac{d\gamma}{dt}| = h(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})^{1/2}$. Define **the geodesic distance** as $d(x,y) = \inf_\gamma L(\gamma)$, where infimum is taken for all paths connecting x and y.

EXERCISE: Prove that the **geodesic distance satisfies triangle inequality and defines metric on M.**

EXERCISE: Prove that **this metric induces the standard topology on M.**

EXAMPLE: Let $M = \mathbb{R}^n$, $h = \sum_i dx_i^2$. **Prove that the geodesic distance coincides with $d(x,y) = |x-y|$.**

EXERCISE: Using partition of unity, **prove that any manifold admits a Riemannian structure.**

Hermitian structures

DEFINITION: A Riemannia metric h on an almost complex manifold is called **Hermitian** if $h(x,y) = h(Ix, Iy)$.

REMARK: Given any Riemannian metric g on an almost complex manifold, **a Hermitian metric h can be obtained as $h = g + I(g)$, where $I(g)(x,y) = g(I(x), I(y))$.**

REMARK: Let I be a complex structure operator on a real vector space V, and g a Hermitian metric. Then **the bilinear form $\omega(x,y) := g(x, Iy)$ is skew-symmetric.** Indeed, $\omega(x,y) = g(x,Iy) = g(Ix, I^2y) = -g(Ix,y) = -\omega(y,x)$.

DEFINITION: A skew-symmetric form $\omega(x,y)$ is called **an Hermitian form on (V,I)**.

REMARK: In the triple I, g, ω, each element can recovered from the other two.

Conformal structure

DEFINITION: Let h, h' be Riemannian structures on M. These Riemannian structures are called **conformally equivalent** if $h' = fh$, where f is a positive smooth function.

DEFINITION: **Conformal structure** on M is a class of conformal equivalence of Riemannian metrics.

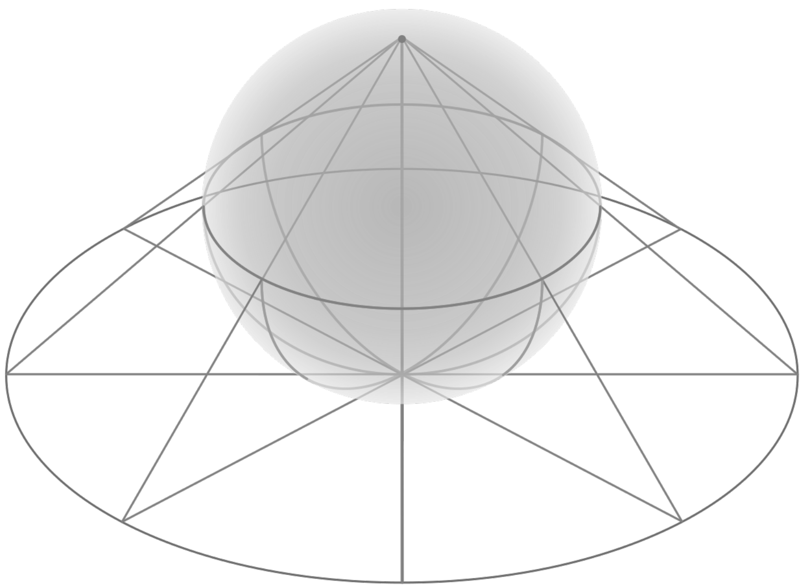
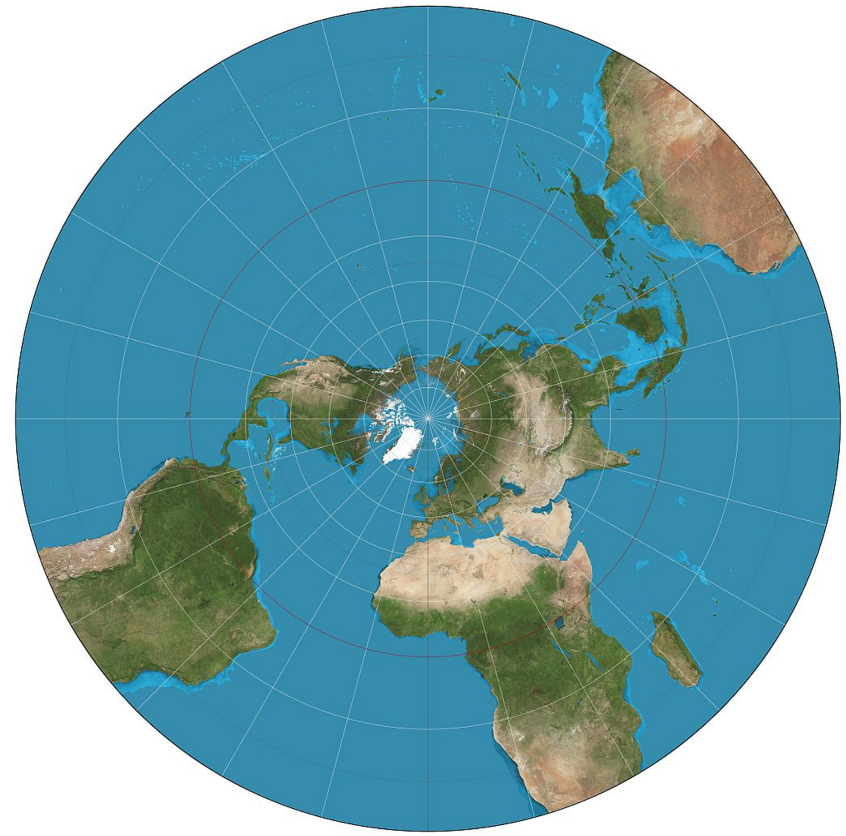
CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Then h and h' are conformally equivalent**. Conversely, any metric conformally equivalent to Hermitian is Hermitian.

Proof: Later today.

REMARK: The last statement is clear from the definition, and true in any dimension **(prove it)**.

To prove that any two Hermitian metrics are conformally equivalent, **we need to consider the standard $U(1)$-action on a complex vector space**.

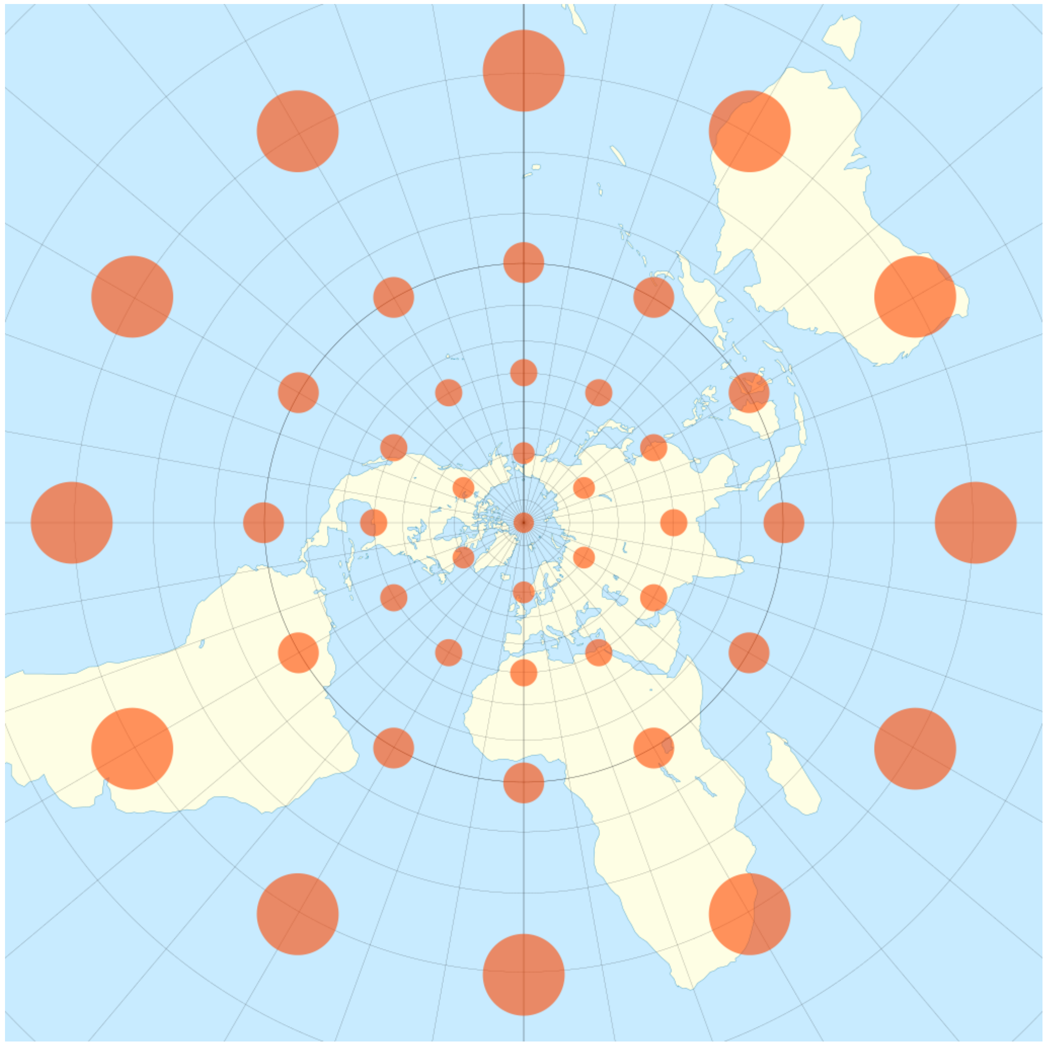
Stereographic projection

Stereographic projection is a light projection from the south pole to a plane tangent to the north pole.

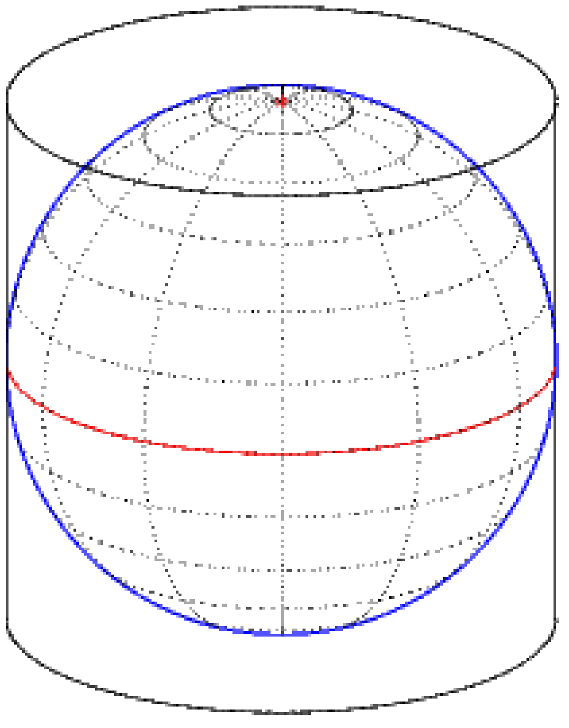
Stereographic projection is conformal (prove it!)

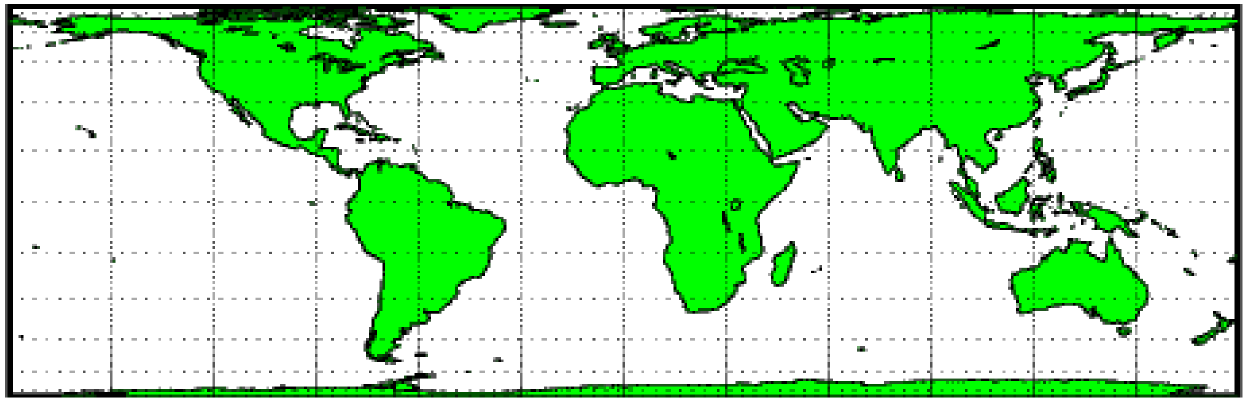
Stereographic projection (2)



The stereographic projection with Tissot's indicatrix of deformation.

Cylindrical projection





Cylindrical projection is not conformal. However, it is volume-preserving.

Standard $U(1)$-action

DEFINITION: Let (V,I) be a real vector space equipped with a complex structure, $U(1)$ the group of unit complex numbers, $U(1) = e^{\sqrt{-1}\,\pi t}$, $t \in \mathbb{R}$. Define the action of $U(1)$ on V as follows: $\rho(t) = e^{tI}$. This is called **the standard $U(1)$-action on a complex vector space**. To prove that this formula defines an action if $U(1) = \mathbb{R}/2\pi\mathbb{Z}$, it suffices to show that $e^{2\pi I} = 1$, which is clear from the eigenvalue decomposition of I.

CLAIM: Let (V,I,h) be a Hermitian vector space, and $\rho:\ U(1) \longrightarrow GL(V)$ the standard $U(1)$-action. **Then h is $U(1)$-invariant.**

Proof: It suffices to show that $\frac{d}{dt}(h(\rho(t)x, \rho(t)x) = 0$. However, $\frac{d}{dt}e^{tI}(x)\Big|_{t=t_0} = I(e^{t_0 I}(x))$, hence

$$\frac{d}{dt}(h(\rho(t)x, \rho(t)x) = h(I(\rho(t)x), \rho(t)x) + h(\rho(t)x, I(\rho(t)x)) = 2\omega(x,x) = 0.$$

■

Hermitian metrics in $\dim_{\mathbb R} = 2$.

COROLLARY: Let h, h' be Hermitian metrics on a space (V,I) of real dimension 2. **Then h and h' are proportional.**

Proof: h and h' are constant on any $U(1)$-orbit. Multiplying h' by a constant, we may assume that $h=h'$ on a $U(1)$-orbit $U(1)x$. Then $h=h'$ everywhere, because **for each non-zero vector $v\in V$, $tv\in U(1)x$ for some $t\in\mathbb R$, giving $h(v,v)=t^{-2}h(tv,tv)=t^{-2}h'(tv,tv)=h'(v,v)$.** ■

DEFINITION: Given two Hermitian forms h,h' on (V,I), with $\dim_{\mathbb R} V=2$, we denote by $\frac{h'}{h}$ a constant t such that $h'=th$.

CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h,h' two Hermitian metrics. **Then h and h' are conformally equivalent**.

Proof: $h'=\frac{h'}{h}h$. ■

EXERCISE: Prove that **Riemannian structure on M is uniquely defined by its conformal class and its Riemannian volume form**.

Conformal structures and almost complex structures

REMARK: The following theorem implies that almost complex structures on a 2-dimensional oriented manifold are equivalent to conformal structures.

THEOREM: Let M be a 2-dimensional oriented manifold. Given a complex structure I, let ν be the conformal class of its Hermitian metric (it is unique as shown above). **Then ν determines I uniquely.**

Proof: Choose a Riemannian structure h compatible with the conformal structure ν. Since M is oriented, the group $SO(2) = U(1)$ acts in its tangent bundle in a natural way: $\rho: U(1) \longrightarrow GL(TM)$. Rescaling h does not change this action, hence it is determined by ν. Now, define I as $\rho(\sqrt{-1})$; then $I^2 = \rho(-1) = -\operatorname{Id}$. Since $U(1)$ acts by isometries, this almost complex structure is compatible with h and with ν. ■

DEFINITION: A Riemann surface is a complex manifold of dimension 1, or (equivalently) an oriented 2-manifold equipped with a conformal structure.

EXERCISE: Prove that **a continuous map from one Riemann surface to another is holomorphic if and only if it preserves the conformal structure**.