Lecture 2: Space forms

Misha Verbitsky

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Homogeneous spaces

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\mathrm{St}_x(G) = \{g \in G \mid g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has $M = G/H$,** where $H = \mathrm{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\mathrm{St}_x(G)^g = \mathrm{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\mathrm{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\mathrm{St}_x(G)$ on T_xM.

DEFINITION: A bilinear symmetric form (or any tensor) Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant bilinear symmetric form (or any tensor) on T_xM, where $M = G/H$ is a homogeneous space. For any $y \in M$ obtained as $y = g(x)$, consider the form Φ_y on T_yM obtained as $\Phi_y := g^*(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \mathrm{St}_x(G)$. Since Φ is h-invariant, **the tensor Φ_y is independent from the choice of g.**

We proved

THEOREM: Let $M = G/H$ be a homogeneous space and $x \in M$ a point. Then the G-invariant bilinear forms (or tensors) on $M = G/H$ **are in bijective correspondence with isotropy invariant bilinear forms (tensors)** on the vector space T_xM. ■

Space forms

DEFINITION: Simply connected space form is a homogeneous Riemannian manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of affine isometries

negative curvature: $SO(1,n)/SO(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

The Riemannian metric is defined in the next slide.

Riemannian metric on space forms

LEMMA: Let $G = SO(n)$ act on $\mathbb{R}^n$ in a natural way. **Then there exists a unique up to a constant multiplier G-invariant symmetric 2-form:** the standard Euclidean metric.

Proof. Step 1: Let g, g' be two metrics. Clearly, it suffices to show that the functions $x \longrightarrow g(x)$ and $x \longrightarrow g'(x)$ are proportional. Fix a vector v on a unit sphere. Replacing g' by $\frac{g(v)}{g'(v)} g'$ if necessary, we can assume that $g = g'$ on a sphere. Indeed, a sphere is an orbit of $SO(n)$, and g, g' are $SO(n)$-invariant.

Step 2: Then $g(\lambda x, \lambda x) = g'(\lambda x, \lambda x)$ for any $x \in S^{n-1}$, $\lambda \in \mathbb{R}$; however, all vectors can be written as λx for appropriate $x \in S^{n-1}$, $\lambda \in \mathbb{R}$. ■

COROLLARY: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Proof: The isotropy group is $SO(n-1)$ in all three cases, and the previous lemma can be applied. ■

Hermitian and conformal structures (reminder)

DEFINITION: Let $h \in \operatorname{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: A Riemannian metric h on an almost complex manifold is called **Hermitian** if $h(x,y) = h(Ix, Iy)$.

DEFINITION: Let h, h' be Riemannian structures on M. These Riemannian structures are called **conformally equivalent** if $h' = fh$, where f is a positive smooth function.

DEFINITION: **Conformal structure** on M is a class of conformal equivalence of Riemannian metrics.

CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Then h and h' are conformally equivalent**. Conversely, any metric conformally equivalent to Hermitian is Hermitian.

Conformal structures and almost complex structures (reminder)

REMARK: The following theorem implies that almost complex structures on a 2-dimensional oriented manifold are equivalent to conformal structures.

THEOREM: Let M be a 2-dimensional oriented manifold. Given a complex structure I, let ν be the conformal class of its Hermitian metric (it is unique as shown above). **Then ν determines I uniquely.**

Proof: Choose a Riemannian structure h compatible with the conformal structure ν. Since M is oriented, the group $SO(2) = U(1)$ acts in its tangent bundle in a natural way: $\rho: U(1) \longrightarrow GL(TM)$. Rescaling h does not change this action, hence it is determined by ν. Now, define I as $\rho(\sqrt{-1})$; then $I^2 = \rho(-1) = -\operatorname{Id}$. Since $U(1)$ acts by isometries, this almost complex structure is compatible with h and with ν. ■

DEFINITION: A Riemann surface is a complex manifold of dimension 1, or (equivalently) an oriented 2-manifold equipped with a conformal structure.

Poincaré-Koebe uniformization theorem

DEFINITION: A **Riemannian manifold of constant curvature** is a Riemannian manifold which is locally isometric to a space form.

THEOREM: (Poincaré-Koebe uniformization theorem) Let M be a Riemann surface. **Then M admits a complete metric of constant curvature in the same conformal class.**

COROLLARY: Any Riemann surface is a quotient of a space form X by a discrete group of isometries $\Gamma \subset \operatorname{Iso}(X)$.

COROLLARY: Any simply connected Riemann surface is conformally equivalent to a space form.

REMARK: We shall prove some cases of the uniformization theorem in later lectures.