Lecture 3: Lie groups

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IMPA, sala 236, 17:00

March 25, 2024

Matrix exponent and Lie groups

DEFINITION: **Exponent** of an endomorphism A is $e^A := \sum_{n=0}^\infty \frac{A^n}{n!}$. **Logarithm** of an endomorphism $1+A$ is $\log(1+A) := \sum_{n=1}^\infty -(-1)^n \frac{A^n}{n}$

EXERCISE: Prove that **exponent is inverse to logarithm in a neighbourhood of 0.**

EXERCISE: Prove that **if $A, B \in \operatorname{End}(V)$ commute, one has $e^{A+B} = e^A e^B$.**

EXERCISE: Find an example when $A, B \in \operatorname{End}(V)$ **do not commute, and $e^{A+B} \neq e^A e^B$.**

EXERCISE: Prove that **exponent is invertible in a sufficiently small neighbourhood of 0** (use the inverse map theorem).

DEFINITION: Let $W \subset \operatorname{End}(V)$ be a subspace obtained by logarithms of all elements in a neighbourhood of zero of a subgroup $G \subset GL(V)$. A group $G \subset GL(V)$ is called **a Lie subgroup of $GL(V)$**, or **a matrix Lie group**, if it is closed and equal to e^W in a neighbourhood of unity. In this case W is called its **Lie algebra**.

REMARK: It is possible to show that **any closed subgroup of $GL(V)$ is a matrix group**. However, for many practical purposes this can be assumed.

Lie groups: first examples

EXAMPLE: From (local) invertibility of exponent it follows that in a neighbourhood of Id_V we have $GL(V) = e^W$, for some $W = \operatorname{End}(V)$ **(prove it)**.

EXERCISE: Prove that $\det e^A = e^{\operatorname{Tr} A}$, where $\operatorname{Tr} A$ is a trace of A.

EXAMPLE: Let $SL(V)$ be the group of all matrices with determinant 1, and $\operatorname{End}_0(V)$ the space of all matrices with trace 0. Then $e^{\operatorname{End}_0(V)} = SL(V)$ **(prove it).** This implies that **$SL(V)$ is also a Lie group.**

Lie groups as submanifolds

DEFINITION: A subset $M \subset \mathbb{R}^n$ is **an m-dimensional smooth submanifold** if for each $x \in M$ there exists an open in $\mathbb{R}^n$ neighbourhood $U \ni x$ and a diffeomorphism from U to an open ball $B \subset \mathbb{R}^n$ which maps $U \cap M$ to an intersection $B \cap R^m$ of B and an m-dimensional linear subspace.

PROPOSITION: Let $G \subset \operatorname{End}(V)$ be a matrix subgroup in $GL(V)$. **Then G is a submanifold.**

Proof. Step 1: From inverse function theorem, it follows that $A \longrightarrow e^A$ is a diffeomorphism on a neighbourhood of 0 mapping the Lie algebra W of G to G.

Step 2: For any $g \in G$, consider the map $x \longrightarrow ge^x$. This map defines a diffeomorphism between a neighbourhood of 0 in $\operatorname{End}(V)$ and a neighbourhood gU of g, mapping W to $gU \subset G$. ■

Orthogonal group as a Lie group

DEFINITION: Let V be a vector space equipped with a non-degenerate bilinear symmetric form h. Then the group of all endomorphisms of V preserving h and orientation is called **the special orthogonal group**, denoted by $SO(V,h)$.

DEFINITION: Consider the space of all $A \in \operatorname{End}(V)$ which satisfy $h(Ax,y) = -h(x,Ay)$. This space is called **the space of antisymmetric matrices** and denoted $\mathfrak{so}(V,h)$.

REMARK: Clearly, $\mathfrak{so}(V,h) = \{A \in \operatorname{End}(V) \ \ | \ \ A^t = -A\}$.

THEOREM: **$SO(V,h)$ is a Lie group, and $\mathfrak{so}(V,h)$ its Lie algebra.**

Proof. Step 1:

$$0 = \frac{d}{dt} h(e^{tA}v, e^{tA}w) = h(Ae^{tA}v, e^{tA}w) + h(e^{tA}v, Ae^{tA}w).$$

If h is e^{tA}-invariant, this gives $0 = h(Av,w) + h(v,Aw)$, hence A is antisymmetric.

Orthogonal group as a Lie group (2)

THEOREM: $SO(V,h)$ **is a Lie group, and** $\mathfrak{so}(V,h)$ **its Lie algebra.**

Proof. Step 1:

$$0 = \frac{d}{dt} h(e^{tA}v, e^{tA}w) = h(e^{tA}A(v), e^{tA}w) + h(e^{tA}v, e^{tA}A(w)).$$

If h is e^{tA}-invariant, this gives $0 = h(Av, w) + h(v, Aw)$, hence A is antisymmetric.

Step 2: Conversely, suppose that A is antisymmetric. Then

$$\frac{d}{dt} h(e^{tA}v, e^{tA}w) = h(Ae^{tA}v, e^{tA}w) + h(e^{tA}v, Ae^{tA}Aw) = 0,$$

hence $h(e^{tA}v, e^{tA}w)$ is independent from t and equal to $h(v,w)$. ■

Classical Lie groups

EXERCISE: Prove that the following groups are Lie groups.

$U(n)$ **("unitary group")**: the group of complex linear automorphisms of $\mathbb{C}^n$ preserving a Hermitian form.

$SU(n)$: **("special unitary group")**: the group of complex linear automorphisms of $\mathbb{C}^n$ of determinant 1 preserving a Hermitian form.

$Sp(2n, \mathbb{R})$ **("symplectic group")**: the group of linear automorphisms of $\mathbb{R}^{2n}$ preserving a non-degenerate, antisymmetric 2-form.

Properties of matrix groups

LEMMA: Let $G \subset GL(V)$ be a matrix Lie group, equal to e^W in a neighbourhood of 1. **Then $W = T_eG \subset \operatorname{End}(V) = T_eGL(V)$.**

Proof: The exponent map $W \longrightarrow e^W \subset G$ is an isomorphism in a neighbourhood of 0, but **the differential of this map is identity.**

LEMMA: Let G be a connected Lie group. **Then G is generated by any neighbourhood of unity.**

Proof: A subgroup $H \subset G$ generated by a given neighbourhood of unity $U \ni e$ is open, The map $U \longrightarrow G$ mapping (u, x) to ux is a diffeomorphism from U to a neighbourhood of x hence it is open. Since any orbit Hx of H acting on G is open, it is also closed, and (unless G is disconnected) there is only one such orbit. ■

Surjective homomorphisms of matrix groups

COROLLARY 1: Let $\psi: G \longrightarrow G'$ be a Lie group homomorphism. Suppose that its differential is surjective. **Then Ψ is surjective on a connected component of unity.**

Proof: Let $W = T_e G$ and $W' = T_e G'$. Since the differential of Ψ is surjective, Ψ is surjective to some neighbourhood of unity by the inverse function theorem. However, a neighbourhood of unity generates G' by the previous lemma. Therefore, Ψ is surjective. ■

COROLLARY 2: Let $\psi: G \longrightarrow G'$ be a Lie group homomorphism. Assume that ψ is injective in a neighbourhood of unity, and $\dim G = \dim G'$. **Then ψ is surjective on a connected component of unity.**

Proof: The differential of ψ is an isomorphism (it is an injective map of vector spaces of the same dimension). Now ψ is surjective by Corollary 1. ■

Group of unitary quaternions

DEFINITION: A quaternion z is called **unitary** if $|z|^2 := z\overline{z} = 1$. The group of unitary quaternions is denoted by $U(1,\mathbb{H})$. **This is a group of all quaternions satisfying $z^{-1} = \overline{z}$.**

CLAIM: Let $\operatorname{im}\mathbb{H} := \mathbb{R}^3$ be the space $aI+bJ+cK$ of all imaginary quaternions. The map $x, y \longrightarrow -\operatorname{Re}(xy)$ defines scalar product on $\operatorname{im}\mathbb{H}$.

CLAIM: This scalar product is positive definite.

Proof: Indeed, if $z = aI + bJ + cK$, $\operatorname{Re}(z^2) = -a^2 - b^2 - c^2$. ■

COROLLARY: Consider the action of $U(1,\mathbb{H})$ on $\operatorname{Im} H$ with $h \in U(1,\mathbb{H})$ mapping $z \in \operatorname{Im}\mathbb{H}$ to $hz\overline{h}$. Since $\overline{hz\overline{h}} = h\overline{z}\overline{h}$, this quaternion also imaginary. Also, $|hz\overline{h}|^2 = hz\overline{h}h\overline{z}\overline{h} = h|z|^2\overline{h} = |z|^2$. **This implies that $U(1,\mathbb{H})$ acts on the space $\operatorname{im}\mathbb{H}$ by isometries.**

DEFINITION: Denote the group of all oriented linear isometries of $\mathbb{R}^3$ by $SO(3)$. This group is called **the group of rotations of $\mathbb{R}^3$.**

REMARK: We have just defined a group homomorphism $U(1,\mathbb{H}) \longrightarrow SO(3)$ mapping h, z to $hz\overline{h}$.

Broom Bridge

"Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication

$$I^2 = J^2 = K^2 = IJK = -1$$

and cut it on a stone of this bridge."

William Rowan Hamilton (watched by his wife)

William Rowan Hamilton (1805 - 1865)

Daniel Doyle's sand sculpture of William Rowan Hamilton (watched by his wife) scratching the formula for his quaternions onto Broom Bridge in Cabra.

$U(\mathbb{H},1)$ is generated by exponents

LEMMA: The group $U(\mathbb{H},1)$ is generated locally by exponents of imaginary quaternions.

Proof: Let h be an imaginary quaternion. Then $\frac{d}{dt}(e^{th}, e^{th}) = (he^{th}, e^{th}) + (e^{th}, he^{th}) = 0$ because $(h(x), y) = -(x, h(y))$ for any imaginary quaternion. Indeed, rescaling h if necessary, we may assume that $h^2 = -1$, then $(h(x), y) = (h^2x, hy) = -(x, hy)$. Since $U(\mathbb{H},1)$ is a 3-sphere, the map $\exp : \operatorname{im}\mathbb{H} \longrightarrow U(\mathbb{H},1)$ is a local diffeomorphism. Since $U(\mathbb{H},1)$ is connected, it is generated by any open neighbourhood of the unity. ■

$SU(2) = U(\mathbb{H}, 1)$

The left action of $U(\mathbb{H}, 1)$ on $\mathbb{H} = \mathbb{C}^2$ commutes with the right action of the algebra $\mathbb{C}$ on $\mathbb{H} = \mathbb{C}^2$. This defines a homomorphism $U(\mathbb{H}, 1) \longrightarrow U(2)$.

THEOREM: This homomorphism **defines an isomorphism $U(\mathbb{H}, 1) \cong SU(2)$,** where $SU(2) \subset U(2)$ is a subgroup of **special unitary matrices** (unitary matrices with determinant 1).

Proof. Step 1: The group $U(2)$ is 4-dimensional, because it is a fixed point set of an anti-complex involution $A \longrightarrow (A^t)^{-1}$ in a space $GL(2, \mathbb{C})$ of real dimension 8. The group $SU(2)$ is a kernel of the determinant map $U(2) \stackrel{\det}{\longrightarrow} U(1)$, **hence it is 3-dimensional.**

Step 2: The map $U(\mathbb{H}, 1) \longrightarrow U(2)$ is by construction injective. Its image is generated by exponents of imaginary quaternions. The elements of $\operatorname{im} \mathbb{H}$ act on $\mathbb{H} = \mathbb{C}^2$ by traceless matrices **(prove this)**. Using the formula $e^{\operatorname{Tr} A} = \det e^A$, we obtain that their exponents have trivial determinant. **This gives an injective map $U(\mathbb{H}, 1) \longrightarrow SU(2)$.** It is surjective by Corollary 2. ■

Group of rotations of $\mathbb{R}^3$

Similar to complex numbers which can be used to describe rotations of $\mathbb{R}^2$, quaternions can be used to describe rotations of $\mathbb{R}^3$.

THEOREM: Let $U(1,\mathbb{H})$ be the group of unitary quaternions acting on $\mathbb{R}^3 = \operatorname{Im} H$ as above: $h(x) := hx\bar{h}$. Then **the corresponding group homomorphism defines an isomorphism $\Psi : U(1,\mathbb{H})/\{\pm 1\} \tilde{\longrightarrow} SO(3)$.**

Proof. Step 1: First, any quaternion h which lies in the kernel of the homomorpism $U(1,\mathbb{H}) \longrightarrow SO(3)$ commmutes with all imaginary quaternions, Such a quaternion must be real **(check this).** Since $|h| = 1$, we have $h = \pm 1$. **This implies that Ψ is injective.**

Step 2: These groups are 3-dimensional. **Then Ψ is surjective by Corollary 2.**

COROLLARY: The group $SO(3)$ is identified with the real projective space $\mathbb{R}P^3$.

Proof: Indeed, $U(1,\mathbb{H})$ is identified with a 3-sphere, and $\mathbb{R}P^3 := S^3/\{\pm 1\}$.■

The group $SO(4)$

Consider the following scalar product on $\mathbb{H} = \mathbb{R}^4$: $g(x,y) = \operatorname{Re}(x\overline{y})$. Obviously from its definition, this form is positive definite. Let $U(1,\mathbb{H}) \times U(1,\mathbb{H})$ act on $\mathbb{H}$ as follows: $h_1, h_2, z \longrightarrow h_1 z \overline{h}_2$, with $z \in \mathbb{H}$ and $h_1, h_2 \in U(1,\mathbb{H})$. Clearly, $|h_1 z \overline{h}_2|^2 = h_1 z \overline{h}_2 h_2 \overline{z} \overline{h}_1 = h_1 z \overline{z} \overline{h}_1 = z\overline{z}$, hence **the group $U(1,\mathbb{H}) \times U(1,\mathbb{H})$ acts on $\mathbb{H} = \mathbb{R}^4$ by isometries.** The group $\ker \Psi$ contains a pair $(-1,-1) \subset U(1,\mathbb{H}) \times U(1,\mathbb{H})$. We denote the group generated by $(-1,-1)$ as $\{\pm 1\} \subset U(1,\mathbb{H}) \times U(1,\mathbb{H})$.

THEOREM: Denote by $SO(4)$ the group of linear orthogonal automorphisms of $\mathbb{R}^4$, and let $\Psi : U(1,\mathbb{H}) \times U(1,\mathbb{H})/\{\pm 1\} \longrightarrow SO(4)$ be the group homomorphism constructed above, $h_1, h_2(x) = h_1 x \overline{h}_2$. **Then Ψ is an isomorphism**. In particular, **$SO(4)$ is diffeomorphic to $S^3 \times S^3/\{\pm 1\}$.**

Proof. Step 1: Again, let $(h_1, h_2) \in \ker \Psi$. Since $\Psi(h_1,h_2)(1) = 1$, this gives $h_2 = \overline{h}_1 = h_1^{-1}$. However, $h_1 z h_1^{-1} = z$ means that h_1 commutes with z, which implies that h_1 commutes with all quaternions, hence it is real. Then $h_1 = \pm 1$. **This proves injectivity of Ψ.**

Step 2: The group $SO(4)$ is 6-dimensional **(prove it)**, and $U(1,\mathbb{H}) \times U(1,\mathbb{H})$ is also 6-dimensional. Then Ψ is surjective by Corollary 2. ∎

$PU(2) = SO(3)$

DEFINITION: Let $U(2) \subset GL(2, \mathbb{C})$ be the group of unitary matrices, and $PU(2)$ its quotient by the group $U(1)$ of diagonal matrices. It is called **projective unitary group**.

REMARK: $PU(2)$ is a quotient of $SU(2)$ by its center $-\operatorname{Id}$ **(prove it)**. The group $U(2)$ acts on $\mathbb{C}P^1$, its action is factorized through $PU(2)$, and all non-trivial $g \in PU(2)$ act on $\mathbb{C}P^1$ non-trivially **(prove it)**.

THEOREM: **$PU(2)$ is isomorphic to $SO(3)$,** the isotropy group of its action on $\mathbb{C}P^1$ is $U(1)$, and the $U(2)$-invariant metric on $\mathbb{C}P^1$ is isometric to the standard Riemannian metric on S^2.

Proof: As shown above, $PU(2) = \frac{SU(2)}{\pm 1}$, and $SO(3) = \frac{U(1,\mathbb{H})}{\pm 1}$. On the other hand, $SU(2) = U(1, \mathbb{H})$.

An element $a \in SU(2)$ fixing a line $x \in \mathbb{C}P^1$ acts on its orthogonal complement by rotations. Since $\det a = 1$, the angle of this rotation uniquely determines the angle of rotation of a on the line x. Therefore, the isotropy group of $SU(2)$-action on $\mathbb{C}P^1$ is S^1. For $PU(2)$ it is $S^1/\{\pm 1\} = S^1$.

Finally, there exists only one, up to a constant, $SO(3) = PU(2)$-invariant metric on $\frac{SO(3)}{SO(2)} = S^2$. ■

Lie algebra

REMARK: Since **the commutator of left-invariant vector fields is left-invariant,** commutator is well defined on the space of left invariant vector fields A. Commutator is a bilinear, antisymmetric operation $A \times A \longrightarrow A$ which satisfies **the Jacobi identity:**

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$$

DEFINITION: A **Lie algebra** is a vector space A equipped with a bilinear, antisymmetric operation $A \times A \longrightarrow A$ which satisfies the Jacobi identity.

THEOREM: (The main theorem of Lie theory) A simply connected Lie group **is uniquely determined by its Lie algebra.** Every finite-dimensional Lie algebra **is obtained as a Lie algebra of a simply connected Lie group**.

DEFINITION: Adjoint representation of a Lie group is the action of G on its Lie algebra T_eG obtained from the adjoint action of G on itself, $g(x) = gxg^{-1}$.

REMARK: Any matrix Lie group $G \subset GL(V)$, **is generated by exponents of its Lie algebra** Lie(G)**,** and locally in a neighbourhood of zero the exponent map $\exp :$ Lie$(G) \longrightarrow G$ is a diffeomorphism.

Left-invariant vector fields

REMARK: A group acts on itself in three different ways: there is **left action** $g(x) = gx$, **right action** $g(x) = xg^{-1}$, and **adjoint action** $g(x) = gxg^{-1}$,

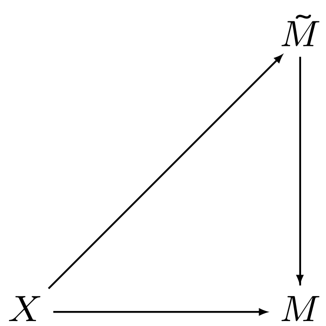
DEFINITION: **Lie algebra** of a Lie group G is the Lie algebra Lie(G) of left-invariant vector fields.

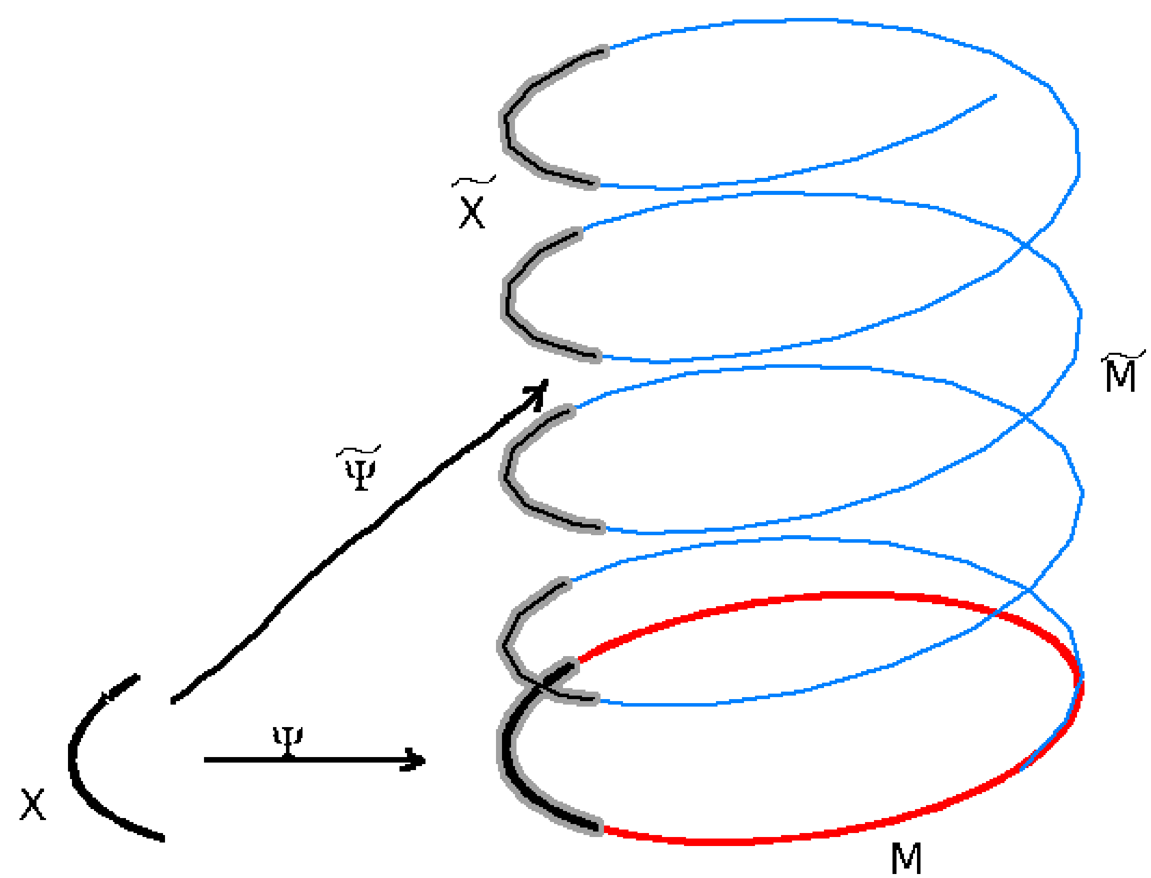
REMARK: Since the group acts on itself freely and transitively, left-invariant vector fields on G are identified T_eG. Indeed, **any vector $x \in T_eG$ can be extended to a left-invariant vector field in a unique way.**

REMARK: **The same is true for any left-invariant tensor on G:** it can be obtained in a unique way from a tensor on a vector space T_eG.

Homotopy lifting principle

THEOREM: (homotopy lifting principle) Let X be a simply connected, locally path connected topological space, and $\tilde{M} \longrightarrow M$ a covering map. Then for each continuous map $X \longrightarrow M$, there exists a lifting $X \longrightarrow \tilde{M}$ making the following diagram commutative.

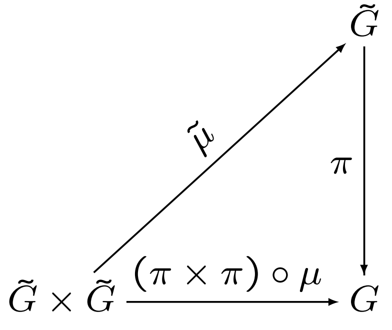


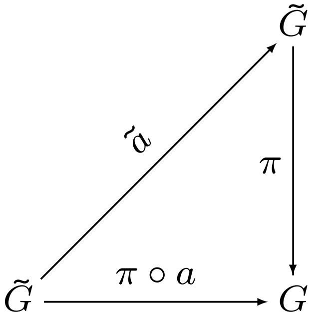


Universal covering of a Lie group

THEOREM: Let G be a connected Lie group, and $\tilde{G}$ its universal covering. **Then $\tilde{G}$ has a unique structure of a Lie group, such that the covering map $\pi: \tilde{G} \longrightarrow G$ is a homomorphism.**

Proof: The multiplication map $\tilde{G} \longrightarrow \tilde{G} \xrightarrow{\tilde{\mu}} \tilde{G}$ is a lifting of the composition of π and multiplication $\tilde{G} \times \tilde{G} \xrightarrow{\pi\times\pi} G \times G \xrightarrow{\mu} G$ mapping the unity $\tilde{e} \times \tilde{e}$ to $\tilde{e}$. Similarly, the inverse map $\tilde{a}: \tilde{G} \longrightarrow \tilde{G}$ is a lifting of the inverse $a: G \longrightarrow G$ mapping $\tilde{e}$ to $\tilde{e}$





Uniqueness and group identities on $\tilde{G}$ both follow from the uniqueness of the homotopy lifting. ■

Classification of 1-dimensional Lie groups

Exercise 1: Prove that **any non-trivial discrete subgroup of $\mathbb{R}$ is cyclic** (isomorphic to $\mathbb{Z}$).

THEOREM: Any 1-dimensional connected Lie group G is isomorphic to S^1 or $\mathbb{R}$.

Proof. Step 1: Any 1-dimensional manifold is diffeomorphic to S^1 or $\mathbb{R}$. By Exercise 1 it suffices to prove that any simply connected, connected 1-dimensional Lie group is isomorphic to $\mathbb{R}$.

Classification of 1-dimensional Lie groups (2)

THEOREM: Any 1-dimensional connected Lie group G is isomorphic to S^1 or $\mathbb{R}$.

Step 2: Since G is simply connected, it is diffeomorphic to $\mathbb{R}$. Let $v \in T_eG$ be a non-zero tangent vector, $\vec{v} \in TG$ the corresponding left-invariant vector field, and $E: \mathbb{R} \longrightarrow G$ a solution of the ODE

$$\frac{d}{dt}E(t) = \vec{v}. \quad (*)$$

mapping 0 to e. A solution of (*), considered as a map from $\mathbb{R}$ to $G = \mathbb{R}$, exists and is uniquely determined by $P(0)$ by the uniqueness and existence of solutions of ODE. Since the left action L_g of G on itself preserves $\vec{v}$, it maps solutions of (*) to solutions of (*). Let $g = E(s)$. Then $t \longrightarrow L_{g^{-1}}E(s+t)$ is a solution of (*) which maps 0 to $E(s)^{-1}E(s) = e$, hence $L_{g^{-1}}E(s+t) = E(t)$ and $E(s+t) = E(s)E(t)$. Therefore, the map $E: \mathbb{R} \longrightarrow G$ is a group homomorphism.

Step 3: Differential of E is non-degenerate, hence E is locally a diffeomorphism; since G is connected, G is generated by a neighbourhood of 0, hence E is surjective. If E is not injective, its kernel is discrete, but then $\ker E = \mathbb{Z}$, and G is a circle. Therefore, E is invertible. ■