Lecture 4: Möbius group

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Laurent power series

THEOREM: (Laurent theorem)
Let f be a holomorphic function on an annulus (that is, a ring)

$$R := \{z \mid \alpha < |z| < \beta\}.$$

Then f can be expressed as a Laurent power series $f(z) = \sum_{i\in\mathbb{Z}} z^i a_i$ converging in R.

Proof: Same as Cauchy formula. **Do that as an exercise.** ■

REMARK: This theorem remains valid if $\alpha = 0$ and $\beta = \infty$.

REMARK: A function $\varphi: \mathbb{C}^* \longrightarrow \mathbb{C}$ uniquely determines its Laurent power series. Indeed, the residue of $z^k\varphi$ in 0 is $\sqrt{-1}\, 2\pi a_{-k-1}$.

REMARK: Let $\varphi: \mathbb{C}^* \longrightarrow \mathbb{C}$ be a holomorphic function, and $\varphi = \sum_{i\in\mathbb{Z}} z^i a_i$ its Laurent power series. **Then $\psi(z) := \varphi(z^{-1})$ has Laurent polynomial $\psi = \sum_{i\in\mathbb{Z}} z^{-i} a_i$.**

Complex projective space

DEFINITION: Let $V=\mathbb{C}^n$ be a complex vector space equipped with a Hermitian form h, and $U(n)$ the group of complex endomorphisms of V preserving h. This group is called **the complex isometry group**.

DEFINITION: **Complex projective space** $\mathbb{C}P^n$ is the space of 1-dimensional subspaces (lines) in $\mathbb{C}^{n+1}$.

REMARK: Since the group $U(n+1)$ of unitary matrices acts on lines in $\mathbb{C}^{n+1}$ transitively **(prove it)**, $\mathbb{C}P^n$ **is a homogeneous space,** $\mathbb{C}P^n=\frac{U(n+1)}{U(1)\times U(n)}$**,** where $U(1)\times U(n)$ is a stabilizer of a line in $\mathbb{C}^{n+1}$.

EXAMPLE: $\mathbb{C}P^1$ is S^2.

Homogeneous and affine coordinates on $\mathbb{C}P^n$

DEFINITION: We identify $\mathbb{C}P^n$ with the set of $n+1$-tuples $x_0 : x_1 : ... : x_n$ defined up to equivalence $x_0 : x_1 : ... : x_n \sim \lambda x_0 : \lambda x_1 : ... : \lambda x_n$, for each $\lambda \in \mathbb{C}^*$. This representation is called **homogeneous coordinates**. **Affine coordinates** in the chart $x_k \neq 0$ are are $\frac{x_0}{x_k} : \frac{x_1}{x_k} : ... : 1 : ... : \frac{x_n}{x_k}$. The space $\mathbb{C}P^n$ is a union of $n+1$ affine charts identified with $\mathbb{C}^n$, with the complement to each chart identified with $\mathbb{C}P^{n-1}$.

CLAIM: Complex projective space is a complex manifold, with the atlas given by affine charts $\mathbb{A}_k = \left\{\frac{x_0}{x_k} : \frac{x_1}{x_k} : ... : 1 : ... : \frac{x_n}{x_k}\right\}$, and the transition functions mapping the set

$$\mathbb{A}_k \cap \mathbb{A}_l = \left\{\frac{x_0}{x_k} : \frac{x_1}{x_k} : ... : 1 : ... : \frac{x_n}{x_k} \;\middle|\; x_l \neq 0\right\}$$

to

$$\mathbb{A}_l \cap \mathbb{A}_k = \left\{\frac{x_0}{x_l} : \frac{x_1}{x_l} : ... : 1 : ... : \frac{x_n}{x_l} \;\middle|\; x_k \neq 0\right\}$$

as a multiplication of all terms by the scalar $\frac{x_k}{x_l}$.

Affine coordinates on $\mathbb{C}P^1$

DEFINITION: We identify $\mathbb{C}P^1$ with the set of pairs $x : y$ defined up to equivalence $x : y \sim \lambda x : \lambda y$, for each $\lambda \in \mathbb{C}^*$. This representation is called **homogeneous coordinates**. **Affine coordinates** are $1 : z$ for $x \neq 0$, $z = y/x$ and $z : 1$ for $y \neq 0$, $z = x/y$. The corresponding gluing functions are given by the map $z \longrightarrow z^{-1}$.

DEFINITION: **Meromorphic function** is a quotient f/g, where f, g are holomorphic and $g \neq 0$.

REMARK: A holomorphic map $\mathbb{C} \longrightarrow \mathbb{C}P^1$ is the same as a pair of maps $f : g$ up to equivalence $f : g \sim fh : gh$. **In other words, holomorphic maps $\mathbb{C} \longrightarrow \mathbb{C}P^1$ are identified with meromorphic functions on $\mathbb{C}$.**

REMARK: In homogeneous coordinates, **an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C})$ acts as $x : y \longrightarrow ax + by : cx + dy$.** Therefore, **in affine coordinates it acts as $z \longrightarrow \frac{az+b}{cz+d}$.**

Möbius transforms

DEFINITION: **Möbius transform** is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: **The group $PGL(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphically.**

The following theorem will be proven later in this lecture.

THEOREM: **The natural map from $PGL(2,\mathbb{C})$ to the group of Möbius transforms is an isomorphism**.

Claim 1: Let $\varphi: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a holomorphic automorphism, $\varphi_0: \mathbb{C} \longrightarrow \mathbb{C}P^1$ its restriction to the chart $z:1$, and $\varphi_\infty: \mathbb{C} \longrightarrow \mathbb{C}P^1$ its restriction $1:z$. We consider φ_0, φ_∞ as meromorphic functions on $\mathbb{C}$. **Then $\varphi_\infty = \varphi_0(z^{-1})^{-1}$.**

Proof: Passing from the coordinate $z:1$ to $z:1$ takes z to z^{-1}. The function φ_∞ is obtained from φ_0 by doing this coordinate change on the domain and on the range. ■

Möbius transforms and $PGL(2,\mathbb{C})$

THEOREM: The natural map from $PGL(2,\mathbb{C})$ to the group $\operatorname{Aut}(\mathbb{C}P^1)$ of Möbius transforms is an isomorphism.

Proof. Step 1: Let $\varphi \in \operatorname{Aut}(\mathbb{C}P^1)$. Since $PSL(2,\mathbb{C})$ acts transitively on pairs of points $x \neq y$ in $\mathbb{C}P^1$, by composing φ with an appropriate element in $PGL(2,\mathbb{C})$ we can assume that $\varphi(0) = 0$ and $\varphi(\infty) = \infty$. This means that we may consider the restrictions φ_0 and φ_∞ of φ to the affine charts as a holomorphic functions on these charts, $\varphi_0, \varphi_\infty : \mathbb{C} \longrightarrow \mathbb{C}$.

Step 2: Let $\varphi_0 = \sum_{i>0} a_i z^i$, $a_1 \neq 0$. Claim 1 gives

$$\varphi_\infty(z) = \varphi_0(z^{-1})^{-1} = a_1 z(1 + \sum_{i\geqslant 2} \frac{a_i}{a_1} z^{-i})^{-1}.$$

Unless $a_i = 0$ for all $i \geqslant 2$, this Laurent series has singularities in 0 and cannot be holomorphic. **Therefore φ_0 is a linear function**, and it belongs to $PGL(2,\mathbb{C})$. ■

Lemma 1: Let φ be a Möbius transform fixing $\infty \in \mathbb{C}P^1$. **Then $\varphi(z) = az+b$ for some $a, b \in \mathbb{C}$ and all $z = z:1 \in \mathbb{C}P^1$.**

Proof: Let $A \in PGL(2,\mathbb{C})$ be a map acting on $\mathbb{C} = \mathbb{C}P^1 \backslash \infty$ as parallel transport mapping $\varphi(0)$ to 0. Then $\varphi \circ A$ is a Möbius transform which fixes ∞ and 0. As shown in Step 2 above, it is a linear function. ■

Conformal automorphisms of $\mathbb{C}$

THEOREM: (Riemann removable singularity theorem) Let $f: \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous function which is holomorphic outside of a finite set. **Then f is holomorphic.**

Proof: Do it as an exercise, using the Cauchy formula. ■

THEOREM: All conformal automorphisms of $\mathbb{C}$ can be expressed as $z \longrightarrow az + b$, where a, b are complex numbers, $a \neq 0$.

Proof: Let φ be a conformal automorphism of $\mathbb{C}$. The Riemann removable singularity theorem implies that φ **can be extended to a holomorphic automorphism of $\mathbb{C}P^1$.** Indeed, $\mathbb{C}P^1$ is obtained as a 1-point compactification of $\mathbb{C}$, and any continuous map from $\mathbb{C}$ to $\mathbb{C}$ is extended to a continuous map on $\mathbb{C}P^1$. Now, Lemma 1 implies that $\varphi(z) = az + b$. ■