Lecture 5: Pseudo-Hermitian forms

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Hermitian and pseudo-Hermitian forms

DEFINITION: Let (V, I) be a (real) vector space equipped with a complex structure, and h a bilinear symmetric form. It is called **pseudo-Hermitian** if $h(x, y) = h(Ix, Iy)$.

REMARK: The corresponding quadratic form $x \mapsto h(x, x)$ is sometimes writen as $h(x)$. **One can recover $h(x, y)$ from $h(x)$ as usual: $2h(x, y) = h(x + y) - h(x) - h(y)$.**

REMARK: Often one considers a complex-valued form $h(x, y) + \sqrt{-1}h(x, Iy)$. It is **sesquilinear** as a form on the complex space: $h(\lambda x, y) = \lambda(x, y)$, $h(x, \lambda y) = \overline{\lambda}(x, y)$, for any $\lambda \in \mathbb{C}$, and the imaginary part $\sqrt{-1}\, h(x, Iy)$ is anti-symmetric.

CLAIM: Let (V, I, h) be a pseudo-Hermitian vector space. Consider V as a complex vector space, $\dim_{\mathbb{C}} V = n$. Then there exists a basis $z_1, ..., z_n$ in V such that $h(z_i, z_j) = 0$ for $i \neq j$ (such a basis is called **orthogonal**). Moreover, this basis can be chosen in such a way that $h(z_i, z_i)$ is ± 1 or 0 (such a basis is called **orthonormal**).

Orthonormal basis for a pseudo-Hermitian form

CLAIM: For any pseudo-Hermitian form h on (V,I), **there exists orthonormal basis $z_1, ..., z_n$.**

Proof: Use induction on $\dim V$. If $h = 0$, this claim is clear. Assume that $h \neq 0$. For any $A \subset V$, denote by $A^\perp$ the space $\{x \in V \mid h(x,a) = 0 \;\forall\, a \in A\}$.

Choose any $z_1 \in V$ such that $h(z_1,z_1) \neq 0$, and let $z_1^{\perp,\mathbb{C}} := \langle z_1, I(z_1)\rangle^\perp = z_1^\perp \cap I(z_1)^\perp$. This is a complex vector space which is orthogonal to z_1. It can also be obtained as an orthogonal complement in the complex vector space (V,I) with respect to the sesquilinear form $h(x,y) + \sqrt{-1}\, h(x,Iy)$.

By induction assumption, the space $z_1^{\perp,\mathbb{C}}$ has an orthonormal basis $z_2, ..., z_n$. **Then $z_1, ..., z_n$ is an orthogonal basis in V.** Replacing z_1 by $h(z_1,z_1)^{1/2} z_1$, we obtain an orthonormal basis $z_1, ..., z_n$. ■

Signature of a Hermitian form

REMARK: By Sylvester's law of inertia, the number of z_i such that $h(z_i, z_i) = 1$, $h(z_i, z_i) = -1$ and $h(z_i, z_i) = 0$ **is independent form the choice of an orthonormal basis.**

DEFINITION: Let (V, I, h) be a vector space with non-degenerate Hermitian form, and $z_1, ..., z_n$ an orthonormal basis, $h(z_i, z_i) = 1$ for $i = 1, ...p$ and $h(z_i, z_i) = 1$ for $i = p+1, ..., n$, with $q = n - p$. Then h is called **Hermitian form of signature (p, q)**. The group of complex linear automorphisms preserving h **is denoted $U(p, q)$.**

Normal form for a pair of Hermitian forms

Theorem 1: Let $V = \mathbb{R}^n$, and $h, h' \in \mathrm{Sym}^2 V^*$ be two bilinear symmetric forms, with h positive definite. **Then there exists a basis $x_1, ..., x_n$ which is orthonormal with respect to h, and orthogonal with respect to h'.**

Theorem 1': Let $V = \mathbb{C}^n$, and $h, h' \in \mathrm{Sym}^2 V^*$ be two (pseudo-)Hermitian forms, with h positive definite. **Then there exists a basis $x_1, ..., x_n$ which is orthonormal with respect to h, and orthogonal with respect to h'.**

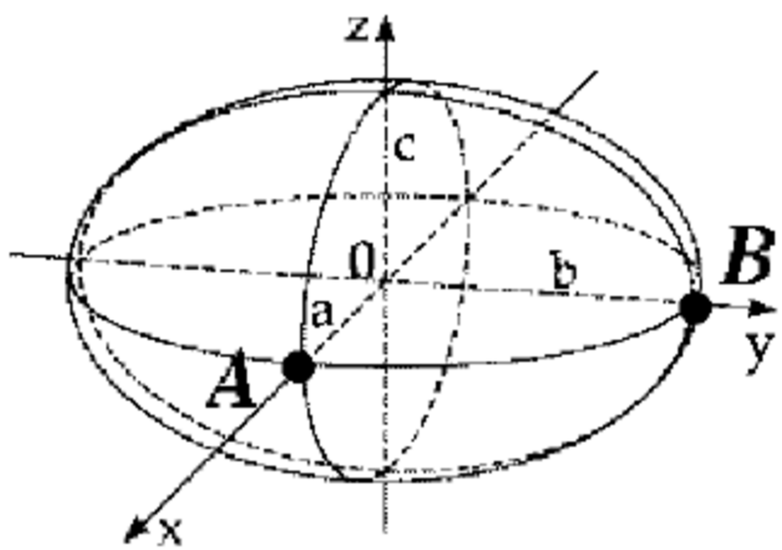
REMARK: In this basis, h' is written as diagonal matrix, with eigenvalues $\alpha_1, ..., \alpha_n$ independent from the choice of the basis. Indeed, consider h, h' as maps from V to V^*, $h(v) = h(v, \cdot)$. Then $h_1 h^{-1}$ is an endomorphism with eigenvalues $\alpha_1, ..., \alpha_n$. **This implies that Theorem 1 gives a normal form of the pair h, h'.**

Finding principal axes of an ellipsoid

REMARK: Theorem 1 implies the following statement about ellipsoids: for any positive definite quadratic form q in $\mathbb{R}^n$, consider the ellipsoid

$$S = \{v \in V \ \ | \ \ q(v) = 1\}.$$

The group $SO(n)$ acts on $\mathbb{R}^n$ preserving the standard scalar product. **Then for some $g \in SO(n)$, $g(S)$ is given by equation $\sum a_i x_i^2 = 1$, where $a_i > 0$.** This is called **finding principal axes of an ellipsoid**.



Maximum of a quadratic form on a sphere

Further on, we prove the following lemma.

LEMMA: Let $V = \mathbb{R}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two bilinear symmetric forms, h positive definite, and $q(v) = h(v,v), q'(v) = h'(v,v)$ the corresponding quadratic forms. Consider q' as a function on a sphere $S = \{v \in V \mid q(v) = 1\}$, and let $x \in S$ be the point where q' attains maximum. Denote by $x^{\perp_h}$ and $x^{\perp_{h'}}$ the orthogonal complement with respect to h, h'. **Then $x^{\perp_h} = x^{\perp_{h'}}$.**

This lemma immediately implies Theorem 1. Let h, h', x as above. Using induction, we may assume that $x^{\perp_h} = x^{\perp_{h'}}$ admits a basis $x_2, ..., x_n$ which is orthonormal for h and orthogonal for h'. **Then $x, x_2, ..., x_n$ is a basis we need.**

Similarly one proves Theorem 1'. Take $x \in S$ as above. Then $I(x)$ is also a maximum for q'. The orthogonal complements to $x, I(x)$ with respect to h and h' coincide by our lemma. Therefore, $W = \langle x, Ix\rangle^{\perp_h} = \langle x, Ix\rangle^{\perp_{h'}}$. We obtain a complex vector space W orthogonal to x with respect to h and h'. Using induction, we find a basis $x_2, ..., x_n$ in W which is orthonormal for h and orthogonal for h'. **Then $x, x_2, ..., x_n$ is such a basis in V.**

Maximum of a quadratic form on a sphere

LEMMA: Let $V = \mathbb{R}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two bilinear symmetric forms, h positive definite, and $q(v) = h(v,v), q'(v) = h'(v,v)$ the corresponding quadratic forms. Consider q' as a function on a sphere $S = \{v \in V \mid q(v) = 1\}$, and let $x \in S$ be the point where q' attains maximum. Denote by $x^{\perp_h}$ and $x^{\perp_{h'}}$ the orthogonal complements with respect to h, h'. **Then $x^{\perp_h} = x^{\perp_{h'}}$.**

Proof: Let us rescale q, q' in such a way that $q \geqslant q'$, with equality on x. Suppose that $v \in x^{\perp_h}$. Then $q(x + \varepsilon v) = q(x) + \varepsilon^2 q(v)$. However, $q'(x + \varepsilon v) = q(x) + \varepsilon^2 q'(v) + 2\varepsilon h'(v,x)$. This gives

$$q(x) + \varepsilon^2 q(v) \geqslant q(x) + \varepsilon^2 q'(v) + 2\varepsilon h'(v,x)$$

cancelling $q(x)$ and dividing by $\varepsilon > 0$, obtain

$$\varepsilon(q(v) - q'(v)) \geqslant 2h'(v,x).$$

for all $\varepsilon > 0$. This implies that $0 \geqslant 2h'(v,x)$ for all $v \in x^{\perp_h}$. **Since $v \mapsto h'(v,x)$ is a linear form on v, inequality $0 \geqslant h'(v,x)$ implies that $h'(v,x) = 0$.** ■