Lecture 6: Circles on a sphere are preserved by the Möbius group

Misha Verbitsky

IMPA, sala 236, 17:00

April 3, 2024

Hermitian and pseudo-Hermitian forms (reminder)

DEFINITION: Let (V, I) be a (real) vector space equipped with a complex structure, and h a bilinear symmetric form. It is called **pseudo-Hermitian** if $h(x, y) = h(Ix, Iy)$.

REMARK: The corresponding quadratic form $x \mapsto h(x, x)$ is sometimes writen as $h(x)$. **One can recover $h(x, y)$ from $h(x)$ as usual: $2h(x, y) = h(x + y) - h(x) - h(y)$.**

CLAIM: Let (V, I, h) be a pseudo-Hermitian vector space. Consider V as a complex vector space, $\dim_{\mathbb{C}} V = n$. Then there exists a basis $z_1, ..., z_n$ in V such that $h(z_i, z_j) = 0$ for $i \neq j$ (such a basis is called **orthogonal**). Moreover, this basis can be chosen in such a way that $h(z_i, z_i)$ is ± 1 or 0 (such a basis is called **orthonormal**).

Theorem 1': Let $V = \mathbb{C}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two (pseudo-)Hermitian forms, with h positive definite. **Then there exists a basis $x_1, ..., x_n$ which is orthonormal with respect to h, and orthogonal with respect to h'.**

REMARK: In this basis, h' is written as diagonal matrix, with eigenvalues $\alpha_1, ..., \alpha_n$ independent from the choice of the basis. Indeed, consider h, h' as maps from V to V^*, $h(v) = h(v, \cdot)$. Then $h_1 h^{-1}$ is an endomorphism with eigenvalues $\alpha_1, ..., \alpha_n$. **This implies that Theorem 1 gives a normal form of the pair h, h'.**

Möbius transforms (reminder)

DEFINITION: Möbius transform is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: The group $PGL(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphically.

The following theorem was proven in Lecture 4.

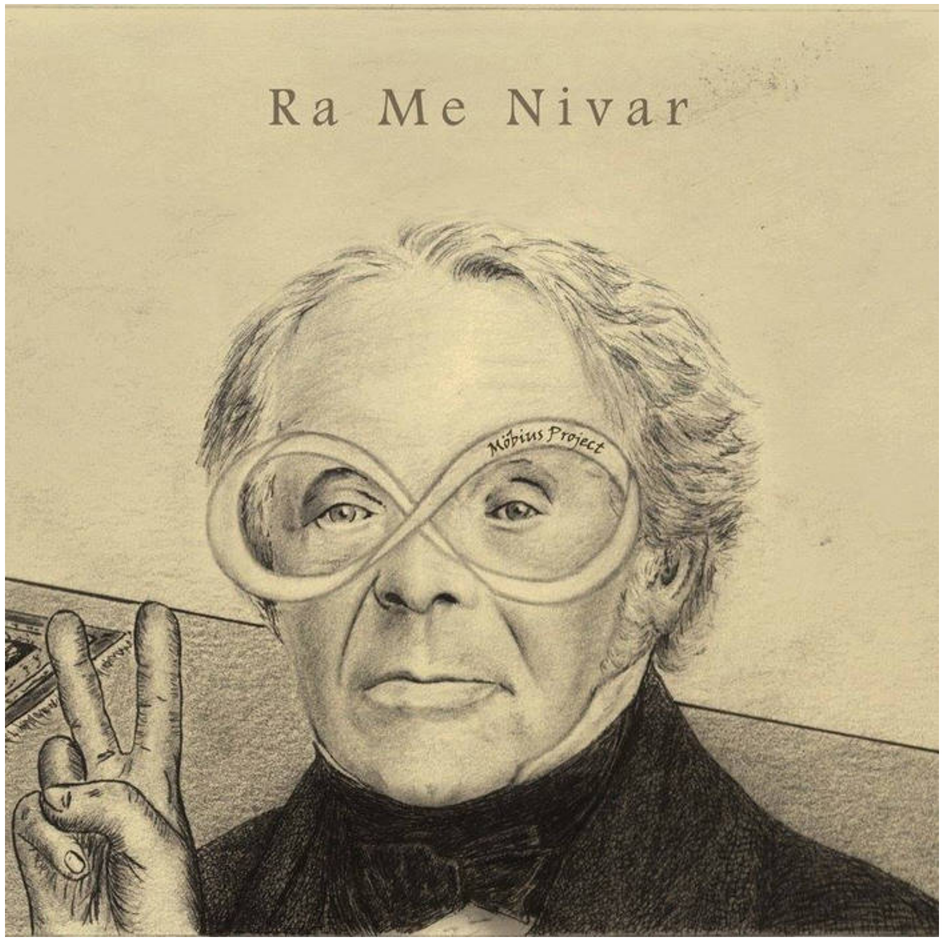
THEOREM: The natural map from $PGL(2,\mathbb{C})$ to the group $\operatorname{Aut}(\mathbb{C}P^1)$ of Möbius transforms is an isomorphism.

August Ferdinand Möbius (1790-1868)

August Ferdinand Möbius (1790-1868)

...Möbius was born in Schulpforta, Electorate of Saxony, and was descended on his mother's side from religious reformer Martin Luther. He studied astronomy at the University of Leipzig under the mathematician and astronomer Karl Mollweide. In 1813, he began to study astronomy under mathematician Carl Friedrich Gauss at the University of Göttingen, while Gauss was the director of the Göttingen Observatory. From there, he went to study with Carl Gauss's instructor, Johann Pfaff, at the University of Halle, where he completed his doctoral thesis "The occultation of fixed stars" in 1815. In 1816, he was appointed as Extraordinary Professor to the "chair of astronomy and higher mechanics" at the University of Leipzig. Möbius died in Leipzig in 1868 at the age of 77.

August Ferdinand Möbius (1790-1868)



Möbius Project, the cover of "Ra Me Nivar" (2014, Italy)

$PU(2) = SO(3)$ (reminder)

DEFINITION: Let $U(2) \subset GL(2,\mathbb{C})$ be the group of unitary matrices, and $PU(2)$ its quotient by the group $U(1)$ of diagonal matrices. It is called **projective unitary group**.

REMARK: $PU(2)$ is a quotient of $SU(2)$ by its center $-\mathrm{Id}$ **(prove it)**. The group $U(2)$ acts on $\mathbb{C}P^1$, its action is factorized through $PU(2)$, and all non-trivial $g \in PU(2)$ act on $\mathbb{C}P^1$ non-trivially **(prove it)**.

THEOREM: **$PU(2)$ is isomorphic to $SO(3)$,** the isotropy group of its action on $\mathbb{C}P^1$ is $U(1)$, and the $U(2)$-invariant metric on $\mathbb{C}P^1$ is isometric to the standard Riemannian metric on S^2.

Circles on a sphere

DEFINITION: A circle in S^2 is an orbit of rotation subgroup, that is, a subgroup $U \subset SO(3) = PU(2) \subset PGL(2,\mathbb{C})$ isomorphic to S^1 and acting on $S^2 = \mathbb{C}P^1$ by isometries.

REMARK: Let U be a rotation group rotating S^2 around an axis passing through x and $y \in S^2$. **Any orbit C of U satisfies $d(x,v) = const$ for all $v \in C$.**

LEMMA: Let z_1, z_2 be a basis in $V = \mathbb{C}^2$, and $h(az_1 + bz_2) = \alpha|a|^2 - \beta|b|^2$ a pseudo-Hermitian form, with $\alpha, \beta \geqslant 0$. **Then the set $Z_h = \mathbb{P}\{x \in V \ \mid \ h(x) = 0\}$ is a circle in $\mathbb{C}P^1$, and all circles can be obtained this way.**

Proof: In homogeneous coordinates, Z_h is the set of all $x : y$ such that $\alpha|x|^2 = \beta|y|^2$, and rotation acts as $x : y \longrightarrow x : e^{\sqrt{-1}\,\theta} y$. Clearly, the orbits of rotation are precisely the sets Z_h for different α, β. ■

Properties of Möbius transform

PROPOSITION: The action of $PGL(2,\mathbb{C})$ on $\mathbb{C}P^1$ maps circles to circles.

Proof. Step 1: Consider a pseudo-Hermitian form h on $V = \mathbb{C}^2$ of signature (1,1). There exists an orthonormal basis $z_1, z_2 \in V$ such that $h(az_1 + bz_2) = \alpha|a|^2 - \beta|b|^2$ with $\alpha, \beta > 0$ real numbers. The set $\{z \mid h(z,z) = 0\}$ is invariant under the rotation $z_1, z_2 \longrightarrow e^{-\sqrt{-1}\,\theta} z_1, e^{\sqrt{-1}\,\theta} z_2$, hence **it is a circle.**

Step 2: By the previous lemma, all circles are obtained this way.

Step 3: $PGL(2,\mathbb{C})$ maps pseudo-Hermitian forms to pseudo-Hermitian forms of the same signature, and therefore **preserves circles.** ∎

Orbits of compact one-parametric subgroups in $PGL(2,\mathbb{C})$

LEMMA: Let $G \cong S^1$ be a compact 1-dimensional subgroup in $PGL(2,\mathbb{C})$. **Then any G-orbit in $\mathbb{C}P^1$ is a circle.**

Proof: Let $V = \mathbb{C}^2$, and consider the natural projection map

$$\pi : SL(V) \longrightarrow PGL(2,\mathbb{C}) = SL(V)/\pm 1.$$

Then $\tilde{G} = \pi^{-1}(G)$ is compact. Chose a $\tilde{G}$-invariant Hermitian metric h on V by averaging a given metric with $\tilde{G}$-action. By definition, circles on $\mathbb{C}P^1$ are orbits of rotation subgroups in $SU(V,h)$. **Since $u(\tilde{G})$ is a 1-dimensional compact subgroup in $SU(V,h)$, its orbit is a circle.** ■