Lecture 7: Isometries of the Poincaré plane

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IMPA, sala 236, 17:00

April 8, 2024

Surjective homomorphisms of matrix groups (reminder)

COROLLARY 1: Let $\psi : G \longrightarrow G'$ be a Lie group homomorphism. Suppose that its differential is surjective. **Then Ψ is surjective on a connected component of unity.**

Proof: Let $W = T_e G$ and $W' = T_e G'$. Since the differential of Ψ is surjective, Ψ is surjective to some neighbourhood of unity by the inverse function theorem. However, a neighbourhood of unity generates G' (Lecture 3). Therefore, Ψ is surjective. ■

COROLLARY 2: Let $\psi : G \longrightarrow G'$ be a Lie group homomorphism. Assume that ψ is injective in a neighbourhood of unity, and $\dim G = \dim G'$. **Then ψ is surjective on a connected component of unity.**

Proof: The differential of ψ is an isomorphism (it is an injective map of vector spaces of the same dimension). Now ψ is surjective by Corollary 1. ■

Some low-dimensional Lie group isomorphisms

DEFINITION: Lie algebra of a Lie group G is the Lie algebra Lie(G) of left-invariant vector fields. **Adjoint representation** of G is the standard action of G on Lie(G). For a Lie group $G = GL(n)$, $SL(n)$, etc., let $PGL(n)$, $PSL(n)$, etc. denote the image of G in GL(Lie(G)) with respect to the adjoint action.

REMARK: This is the same as a quotient G/Z by the center Z of G (prove it).

DEFINITION: Define $SO(1,2)$ as the group of orthogonal matrices on a 3-dimensional real space equipped with a scalar product of signature (1,2), $SO^+(1,2)$ its connected component of unity, and $U(1,1)$ the group of complex linear maps $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ preserving a pseudio-Hermitian form of signature (1,1).

THEOREM: The groups $PU(1,1)$, $PSL(2,\mathbb{R})$, $SO^+(1,2)$ are isomorphic.

Proof: Isomorphism $PU(1,1) = SO^+(1,2)$ will be established later today. To see $PSL(2,\mathbb{R}) \cong SO^+(1,2)$, consider **the Killing form** κ on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, $a, b \longrightarrow$ Tr(ab). **Check that it has signature $(1,2)$. Then the image of $SL(2,\mathbb{R})$ in automorphisms of its Lie algebra is mapped to $SO^+(\mathfrak{sl}(2,\mathbb{R}),\kappa) = SO^+(1,2)$.** Both groups are 3-dimensional, hence it is an isomorphism ("Corollary 2"). ■

REMARK: We prove this theorem by showing that **all these groups are isomorphic to the group of conformal automorphism of a disk.**

Möbius transforms (reminder)

DEFINITION: Möbius transform is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: The group $PGL(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphially.

THEOREM: The natural map from $PGL(2,\mathbb{C})$ to the group of Möbius transforms is an isomorphism.

DEFINITION: A circle in S^2 is an orbit of a 1-parametric isometric rotation subgroup $U \subset PGL(2,\mathbb{C})$.

PROPOSITION: The action of $PGL(2,\mathbb{C})$ on $\mathbb{C}P^1$ maps circles to circles.

THEOREM: All conformal automorphisms of $\mathbb{C}$ can be expressed by $z \longrightarrow az+b$, where a, b are complex numbers, $a \neq 0$.

Schwarz lemma

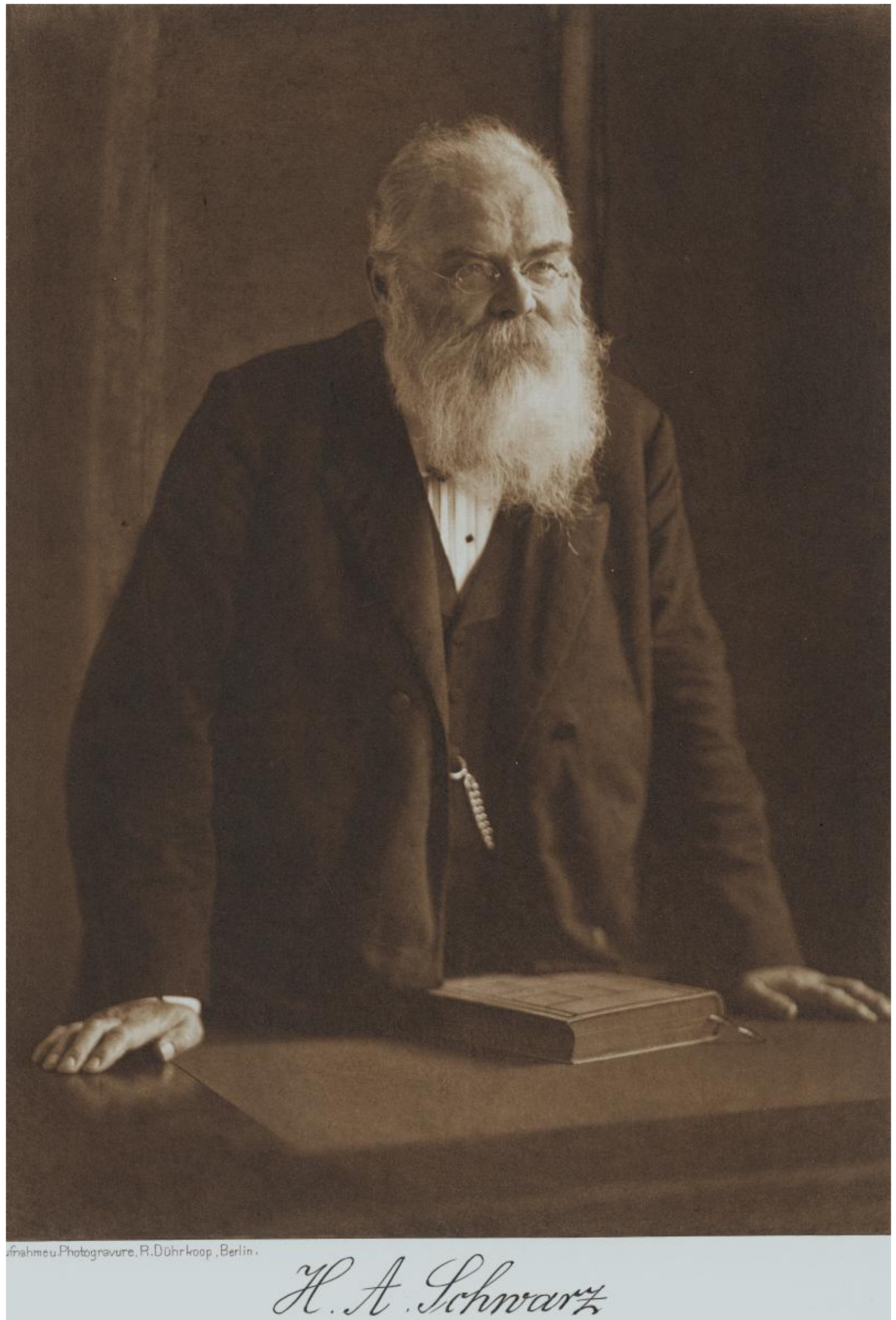
CLAIM: (maximum principle) Let f be a holomorphic function defined on an open set U. **Then f cannot have strict maxima in U. If f has non-strict maxima, it is constant.**

Proof: By Cauchy formula, $f(0) = \frac{1}{2\pi}\int_{\partial\Delta} f(z)\frac{dz}{-\sqrt{-1}\, z}$, where Δ is a disk in $\mathbb{C}$. An elementary calculation gives $\frac{dz}{-\sqrt{-1}\, z}\big|_{\partial\Delta} = \operatorname{Vol}(\partial\Delta)$ – the volume form on $\partial\Delta$. Therefore, $f(0)$ is the average of $f(z)$ on the circle, and it is the average of $f(z)$ on the disk Δ. Now, absolute value of the average $|\operatorname{Av}_{x\in S}\mu(x)|$ of a complex-valued function μ on a set S is equal to $\max_{x\in S}|\mu(s)|$ only if $\mu = const$ almost everywhere on S **(check this)**. ■

LEMMA: (Schwarz lemma) Let $f : \Delta \longrightarrow \Delta$ be a map from disk to itself fixing 0. **Then $|f'(0)| \leqslant 1$, and equality can be realized only if $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$.**

Proof: Consider the function $\varphi := \frac{f(z)}{z}$. Since $f(0) = 0$, it is holomorphic, and since $f(\Delta) \subset \Delta$, on the boundary $\partial\Delta$ we have $|\varphi||_{\partial\Delta} \leqslant 1$. Now, **the maximum principle implies that $|f'(0)| = |\varphi(0)| \leqslant 1$,** and equality is realized only if $\varphi = const$. ■

Karl Hermann Amandus Schwarz (1843-1921)



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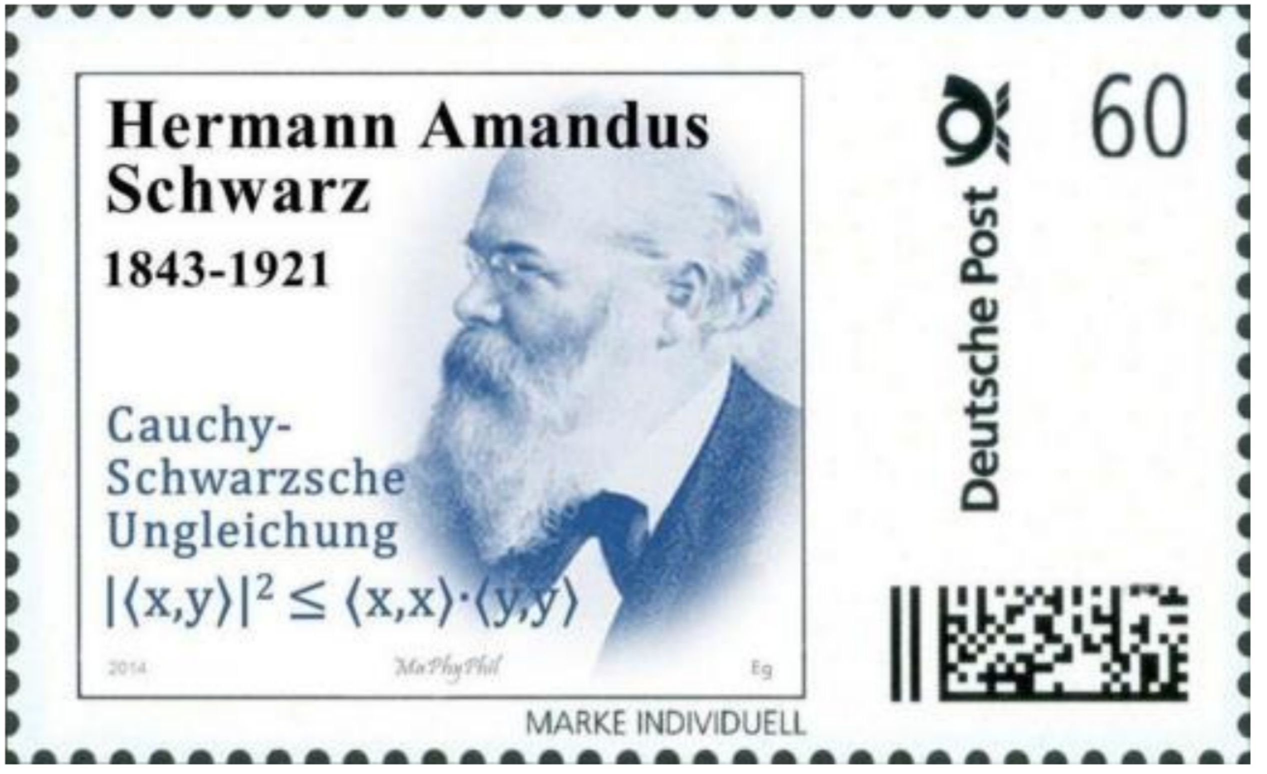

Schwarz was born in Hermsdorf, Silesia (now Sobieszów, Poland). In 1868 he married Marie Kummer, who was the daughter to the mathematician Ernst Eduard Kummer and Ottilie née Mendelssohn (a daughter of Nathan Mendelssohn's and granddaughter of Moses Mendelssohn). Schwarz originally studied chemistry in Berlin, but Ernst Eduard Kummer and Karl Theodor Wilhelm Weierstrass persuaded him to change to mathematics. He received his Ph.D. from the Universität Berlin in 1864 and was advised by Kummer and Weierstrass. In 1892 he became a member of the Berlin Academy of Science and a professor at the University of Berlin, where his students included Paul Koebe and Ernst Zermelo.

Conformal automorphisms of the disk act transitively

CLAIM: Let $\Delta \subset \mathbb{C}$ be the unit disk. **Then the group $\operatorname{Aut}(\Delta)$ of its holomorphic automorphisms acts on Δ transitively.**

Proof. Step 1: Let $V_a(z) = \frac{z-a}{1-\overline{a}z}$ for some $a \in \Delta$. Then $V_a(0) = -a$. To prove transitivity, it remains to show that $V_a(\Delta) = \Delta$.

Step 2: For $|z| = 1$, we have

$$|V_a(z)| = |V_a(z)||z| = \left|\frac{z\overline{z} - a\overline{z}}{1 - \overline{a}z}\right| = \left|\frac{1 - a\overline{z}}{1 - \overline{a}z}\right| = 1.$$

Therefore, V_a preserves the circle. Maximum principle implies that V_a maps its interior to its interior.

Step 3: To prove invertibility, we interpret V_a as an element of the Möbius group $PGL(2, \mathbb{C})$. ■

Transitive action is determined by a stabilizer of a point

Lemma 2: Let $M = G/H$ be a homogeneous space, and $\Psi : G_1 \longrightarrow G$ a homomorphism such that G_1 acts on M transitively and $\mathrm{St}_x(G_1) = \mathrm{St}_x(G)$. **Then $G_1 = G$.**

Proof: Since any element in $\ker \Psi$ belongs to $\mathrm{St}_x(G_1) = \mathrm{St}_x(G) \subset G$, the homomorphism Ψ is injective. It remais only to show that Ψ is surjective.

Let $g \in G$. Since G_1 acts on M transitively, $gg_1(x) = x$ for some $g_1 \in G_1$. Then $gg_1 \in \mathrm{St}_x(G_1) = \mathrm{St}_x(G) \subset \operatorname{im} G_1$. This gives $g \in G_1$. ■

Group of conformal automorphisms of the disk

REMARK: The group $PU(1,1) \subset PGL(2,\mathbb{C})$ of unitary matrices preserving a pseudo-Hermitian form h of signature (1,1) acts on a disk $\{l \in \mathbb{C}P^1 \ | \ h(l,l) > 0\}$ by holomorphic automorphisms.

COROLLARY: Let $\Delta \subset \mathbb{C}$ be the unit disk, Aut(Δ) the group of its conformal automorphisms, and $\Psi: \ PU(1,1) \longrightarrow$ Aut(Δ) the map constructed above. **Then Ψ is an isomorphism.**

Proof: We use Lemma 2. Both groups act on Δ transitively, hence **it suffices only to check that $\operatorname{St}_x(PU(1,1)) = S^1$ and $\operatorname{St}_x(\operatorname{Aut}(\Delta)) = S^1$.** The first isomorphism is clear, because the space of unitary automorphisms fixing a vector v is $U(v^\perp)$. The second isomorphism follows from Schwarz lemma **(prove it!)**. ■

COROLLARY: Let h be a homogeneous metric on $\Delta = PU(1,1)/S^1$. **Then (Δ, h) is conformally equivalent to $(\Delta, \text{flat metric})$.**

Proof: The group $\operatorname{Aut}(\Delta) = PU(1,1)$ acts on Δ holomorphically, that is, preserving the conformal structure of the flat metric. However, homogeneous conformal structure on $PU(1,1)/S^1$ is unique for the same reason the homogeneous metric is unique up to a contant multiplier **(prove it)**. ■

Upper half-plane

REMARK: The map $z \longrightarrow -(z-\sqrt{-1}\,)^{-1} - \frac{\sqrt{-1}}{2}$ induces a diffeomorphism from the unit disc in $\mathbb{C}$ to the upper half-plane $\mathbb{H}^2$ **(prove it)**.

PROPOSITION: The group $\operatorname{Aut}(\Delta)$ acts on the upper half-plane $\mathbb{H}^2$ as $z \xrightarrow{A} \frac{az+b}{cz+d}$, where $a,b,c,d \in \mathbb{R}$, and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$.

REMARK: The group of such A is naturally identified with $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$.

Proof: The group $PSL(2,\mathbb{R})$ preserves the line $\operatorname{im} z = 0$, hence acts on $\mathbb{H}^2$ by conformal automorphisms. The stabilizer of a point is S^1 **(prove it).** Now, Lemma 2 implies that $PSL(2,\mathbb{R}) = PU(1,1)$. ■

COROLLARY: The group of conformal automorphisms of $\mathbb{H}^2$ acts on $\mathbb{H}^2$ preserving a unique, up to a constant, Riemannian metric. **The Riemannian manifold $PSL(2,\mathbb{R})/S^1$ obtained this way is isometric to a hyperbolic space.**