# Lecture 8: Geodesics on a hyperbolic plane

Misha Verbitsky

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## Some low-dimensional Lie group isomorphisms (reminder)

**DEFINITION: Lie algebra** of a Lie group $G$ is the Lie algebra Lie$(G)$ of left-invariant vector fields. **Adjoint representation** of $G$ is the standard action of $G$ on Lie$(G)$. For a Lie group $G = GL(n)$, $SL(n)$, etc., let $PGL(n)$, $PSL(n)$, etc. denote the image of $G$ in $GL(\text{Lie}(G))$ with respect to the adjoint action.

**REMARK: This is the same as a quotient $G/Z$ by the center $Z$ of $G$ (prove it).**
**DEFINITION:** Define $SO(1,2)$ as the group of orthogonal matrices on a 3-dimensional real space equipped with a scalar product of signature (1,2), $SO^+(1,2)$ its connected component of unity, and $U(1,1)$ the group of complex linear maps $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ preserving a pseudio-Hermitian form of signature (1,1).

**THEOREM: The groups $PU(1,1)$, $PSL(2,\mathbb{R})$, $SO^+(1,2)$ are isomorphic.**

**Proof:** Isomorphism $PU(1,1) = SO^+(1,2)$ will be established later today. To see $PSL(2,\mathbb{R}) \cong SO^+(1,2)$, consider **the Killing form** $\kappa$ on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, $a, b \longrightarrow \text{Tr}(ab)$. **Check that it has signature $(1,2)$. Then the image of $SL(2,\mathbb{R})$ in automorphisms of its Lie algebra is mapped to $SO^+(\mathfrak{sl}(2,\mathbb{R}), \kappa) = SO^+(1,2)$.** Both groups are 3-dimensional, hence it is an isomorphism ("Corollary 2"). ■
**REMARK:** We prove this theorem by showing that **all these groups are isomorphic to the group of conformal automorphism of a disk.**

## Möbius transforms (reminder)

**DEFINITION: Möbius transform** is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

**REMARK: The group $PGL(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphially.**

**THEOREM: The natural map from $PGL(2,\mathbb{C})$ to the group of Möbius transforms is an isomorphism**.

**DEFINITION: A circle in $S^2$** is an orbit of a 1-parametric isometric rotation subgroup $U \subset PGL(2,\mathbb{C})$.

**PROPOSITION: The action of $PGL(2,\mathbb{C})$ on $\mathbb{C}P^1$ maps circles to circles.**

**THEOREM: All conformal automorphisms of $\mathbb{C}$ can be expressed by $z \longrightarrow az+b$,** where $a, b$ are complex numbers, $a \neq 0$.

## Transitive action is determined by a stabilizer of a point (reminder)

**Lemma 2:** Let $M = G/H$ be a homogeneous space, and $\Psi : G_1 \longrightarrow G$ a homomorphism such that $G_1$ acts on $M$ transitively and $\mathrm{St}_x(G_1) = \mathrm{St}_x(G)$. **Then $G_1 = G$.**

**Proof:** Since any element in $\ker\Psi$ belongs to $\mathrm{St}_x(G_1) = \mathrm{St}_x(G) \subset G$, the homomorphism $\Psi$ is injective. It remais only to show that $\Psi$ is surjective.

Let $g \in G$. Since $G_1$ acts on $M$ transitively, $gg_1(x) = x$ for some $g_1 \in G_1$. Then $gg_1 \in \mathrm{St}_x(G_1) = \mathrm{St}_x(G) \subset \operatorname{im} G_1$. This gives $g \in G_1$. ■

## Upper half-plane (reminder)

**REMARK:** The map $z \longrightarrow -(z-\sqrt{-1}\,)^{-1} - \frac{\sqrt{-1}}{2}$ induces a diffeomorphism from the unit disc in $\mathbb{C}$ to the upper half-plane $\mathbb{H}^2$ **(prove it)**.

**PROPOSITION: The group $\operatorname{Aut}(\Delta)$ acts on the upper half-plane $\mathbb{H}^2$ as $z \xrightarrow{A} \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$, and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$.**

**REMARK:** The group of such $A$ is naturally identified with $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$.

**Proof:** The group $PSL(2,\mathbb{R})$ preserves the line $\operatorname{im} z = 0$, hence acts on $\mathbb{H}^2$ by conformal automorphisms. The stabilizer of a point is $S^1$ **(prove it).** Now, Lemma 2 implies that $PSL(2,\mathbb{R}) = PU(1,1)$. ■

**COROLLARY:** The group of conformal automorphisms of $\mathbb{H}^2$ acts on $\mathbb{H}^2$ preserving a unique, up to a constant, Riemannian metric. **The Riemannian manifold $PSL(2,\mathbb{R})/S^1$ obtained this way is isometric to the hyperbolic plane $\mathbb{H}^2$. Indeed, $\mathbb{H}^2 = \frac{SO^+(1,2)}{SO(2)}$, and $SO^+(1,2) = PSL(2,\mathbb{R})$**

## Upper half-plane as a Riemannian manifold

**DEFINITION: Poincaré half-plane** is the upper half-plane equipped with a homogeneous metric of constant negative curvature constructed above.

**THEOREM:** Let $(x, y)$ be the usual coordinates on the upper half-plane $\mathbb{H}^2$. **Then the Riemannian form $s \in \operatorname{Sym}^2 T^*\mathbb{H}^2$ is written as $s = const\frac{dx^2+dy^2}{y^2}$.**

**Proof:** Since the complex structure on $\mathbb{H}^2$ is the standard one and all Hermitian structures are conformal (Lecture 1), we obtain that $s = \mu(dx^2 + dy^2)$, where $\mu \in C^\infty(\mathbb{H}^2)$. **It remains to find $\mu$, using the fact that $s$ is $PSL(2, \mathbb{R})$-invariant.**

For each $a \in \mathbb{R}$, the parallel transport $z \longrightarrow z + a$ fixes $s$, hence $\mu$ is a function of $y$. For any $\lambda \in \mathbb{R}^{>0}$, the homothety $H_\lambda(z) = \lambda z$ also fixes $s$; since $\mathbb{H}_\lambda(dx^2 + dy^2) = \lambda^2(dx^2 + dy^2)$, we have $\mu(\lambda z) = \lambda^{-2}\mu(z)$ for any $z \in \mathbb{H}^2$. The only function $\mu(x, y)$ which is constant in $x$ and satisfies $\mu(\lambda y) = \lambda^{-2}\mu(y)$ is $\mu(x, y) = const \cdot y^{-2}$. ■

## Geodesics on Riemannian manifold

**DEFINITION: Minimising geodesic** in a Riemannian manifold is a piecewise smooth path connecting $x$ to $y$ such that its length is equal to the geodesic distance. **Geodesic** is a piecewise smooth path $\gamma$ such that for any $x \in \gamma$ there exists a neighbourhood of $x$ in $\gamma$ which is a minimising geodesic.

**EXERCISE: Prove that a big circle in a sphere is a geodesic. Prove that an interval of a big circle of length $\leqslant \pi$ is a minimising geodesic.**

## Geodesics in Poincaré half-plane

**THEOREM: Geodesics on a Poincaré half-plane are vertical straight lines and their images under the action of $SL(2,\mathbb{R})$.**

**Proof. Step 1:** Let $a, b \in \mathbb{H}^2$ be two points satisfying $\operatorname{Re} a = \operatorname{Re} b$, and $l$ the vertical line connecting these two points. Denote by Π the orthogonal projection from $\mathbb{H}^2$ to the vertical line connecting $a$ to $b$. For any tangent vector $v \in T_z\mathbb{H}^2$, one has $|D\pi(v)| \leqslant |v|$, and the equality means that $v$ is vertical **(prove it)**. Therefore, **a projection of a path $\gamma$ connecting $a$ to $b$ to $l$ has length $\leqslant L(\gamma)$, and the equality is realized only if $\gamma$ is a straight vertical interval.**

**Step 2:** For any points $a, b$ in the Poincaré half-plane, **there exists an isometry mapping $(a, b)$ to a pair of points $(a_1, b_1)$ such that $\operatorname{Re}(a_1) = \operatorname{Re}(b_1)$. (Prove it!)**

**Step 3:** Using Step 2, we prove that **any geodesic $\gamma$ on a Poincaré half-plane is obtained as an isometric image of a straight vertical line:** $\gamma = v(\gamma_0)$, $v \in \operatorname{Iso}(\mathbb{H}^2) = PSL(2,\mathbb{R})$ ■

**COROLLARY:** Any two points in $\mathbb{H}^2$ **are connected by a unique geodesic.**

■

## Geodesics in Poincaré half-plane

**CLAIM:** Let $S$ be a circle or a straight line on a complex plane $\mathbb{C} = \mathbb{R}^2$, and $S_1$ closure of its image in $\mathbb{C}P^1$ inder the natural map $z \longrightarrow 1 : z$. **Then $S_1$ is a circle, and any circle in $\mathbb{C}P^1$ is obtained this way.**

**Proof:** The circle $S_r(p)$ of radius $r$ centered in $p \in \mathbb{C}$ is given by equation $|p - z| = r$, in homogeneous coordinates it is $|px - z|^2 = r|x|^2$. This is the zero set of the pseudo-Hermitian form $h(x, z) = |px - z|^2 - |x|^2$, hence it is a circle. ■

**COROLLARY: Geodesics on the Poincaré half-plane are vertical straight lines and half-circles orthogonal to the line $\operatorname{im} z = 0$ in the intersection points.**

**Proof:** We have shown that geodesics in the Poincaré half-plane are Möbius transforms of straight lines orthogonal to $\operatorname{im} z = 0$. However, any Möbius transform preserves angles and maps circles or straight lines to circles or straight lines. ■

## Poincaré metric on a disk

**DEFINITION: Poincaré metric** on the unit disk $\Delta \subset \mathbb{C}$ is an Aut($\Delta$)-invariant metric **(it is unique up to a constant multiplier; prove it)**.

**DEFINITION:** Let $f: M \longrightarrow M_1$ be a map of metric spaces. Then $f$ is called **$C$-Lipschitz** if $d(x,y) \geqslant Cd(f(x), f(y))$. A map is called **Lipschitz** if it is $C$-Lipschitz for some $C > 0$.

**THEOREM: (Schwarz-Pick lemma)**
**Any holomorphic map $\varphi: \Delta \longrightarrow \Delta$ from a unit disk to itself is 1-Lipschitz with respect to Poincaré metric.**

**Proof. Step 1:** We need to prove that for each $x \in \Delta$ the norm of the differential, taken with respect to the Poincaré metric, satisfies $|D\varphi_x|_P \leqslant 1$. Since the automorphism group acts on $\Delta$ transitively, **it suffices to prove that $|D\varphi_x| \leqslant 1$ when $x = 0$ and $\varphi(x) = 0$.**

**Step 2:** This is Schwarz lemma. ∎

## Space forms (reminder)

**DEFINITION: Simply connected space form** is a homogeneous Riemannian manifold of one of the following types:

**positive curvature:** $S^n$ (an $n$-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

**zero curvature:** $\mathbb{R}^n$ (an $n$-dimensional Euclidean space), equipped with an action of isometries

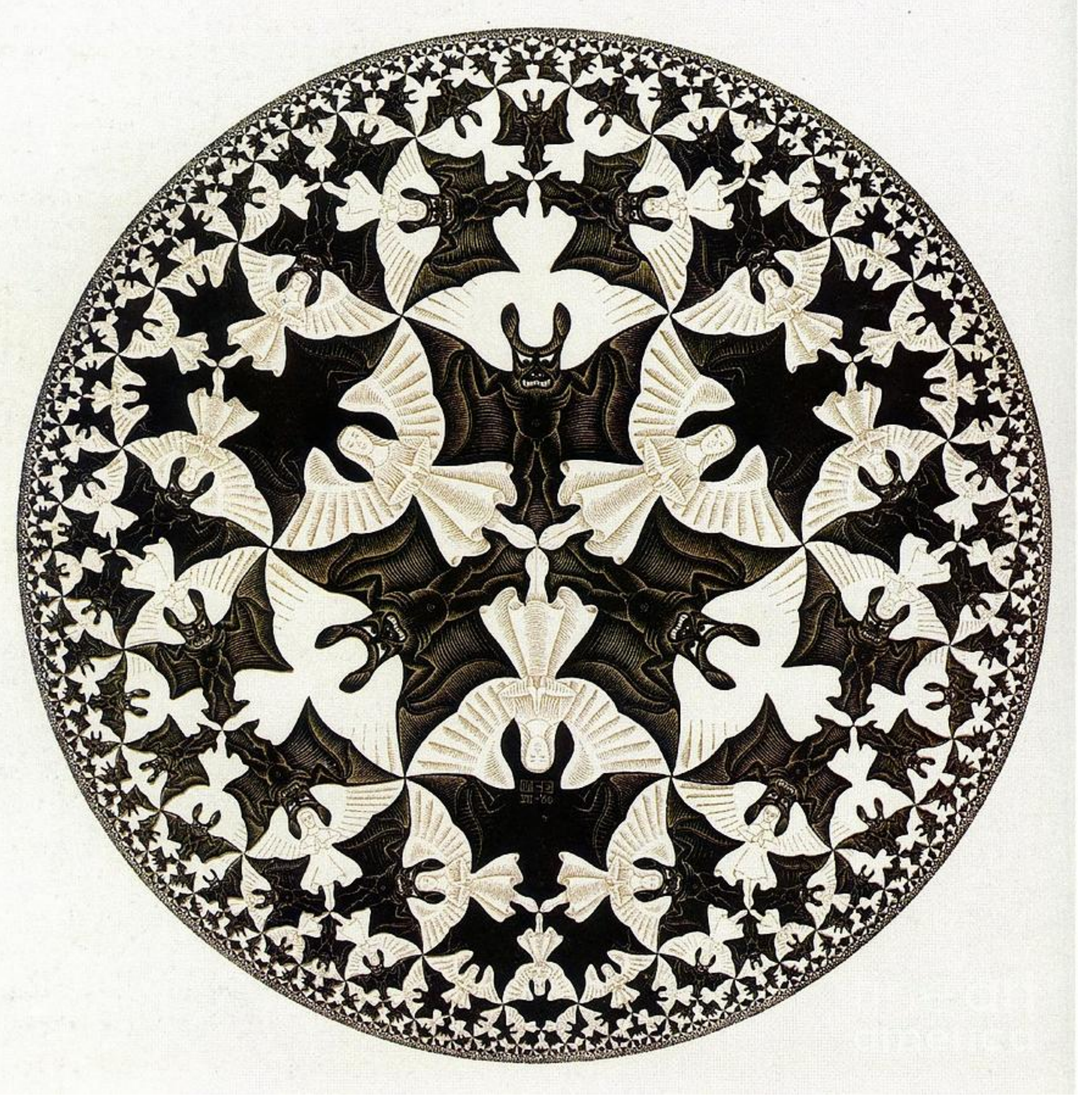
**negative curvature:** $\mathbb{H}^n := SO(1,n)/SO(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

The Riemannian metric is defined by a lemma, proven in Lecture 3.

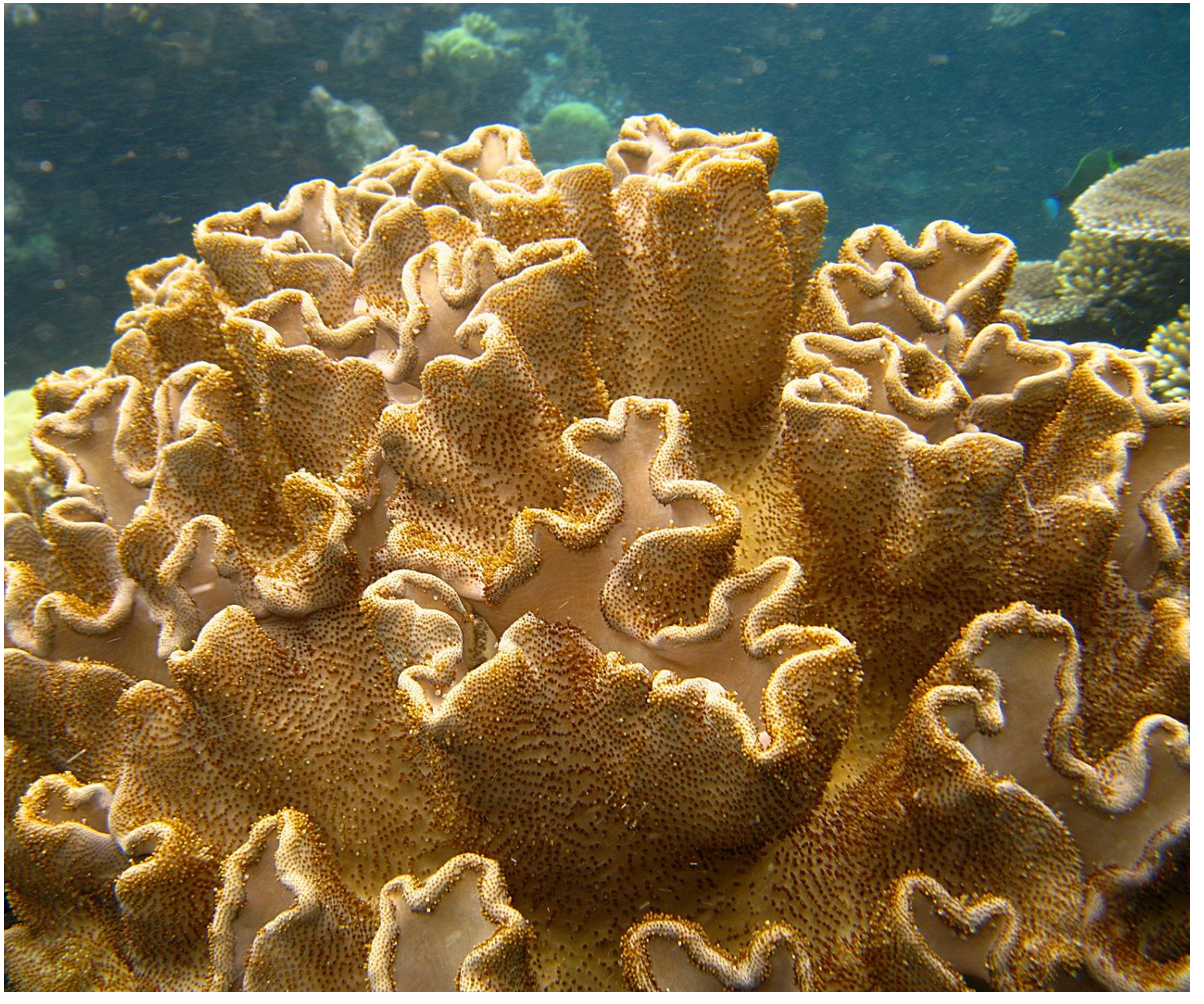
**LEMMA:** Let $M = G/H$ be a simply connected space form. **Then $M$ admits a unique (up to a constant multiplier) $G$-invariant Riemannian form.**

**REMARK: We shall consider space forms as Riemannian manifolds** equipped with a $G$-invariant Riemannian form.

## M. C. Escher, Circle Limit IV



## Crochet coral (Great Barrier Reef, Australia)



## Reflections and geodesics

**DEFINITION: A reflection** on a hyperbolic plane is an involution which reverses orientation and has a fixed set of codimension 1.

**EXAMPLE:** Let the quadratic form $q$ be written as $q(x_1, x_2, x_3) = x_1^2 - x_2^2 - x_3^2$. Then the map $x_1, x_2, x_3 \longrightarrow x_1, x_2, -x_3$ is clearly a reflection.

**CLAIM: Fixed point set of a reflection is a geodesic.** This produces **a bijection between the set of geodesics and the set of reflections.**

**Proof:** Let $x, y \in F$ be two distinct points on a fixed set of a reflection $\tau$. Since the geodesic connecting $x$ and $y$ is unique, it is $\tau$-invariant. Therefore, it is contained in $F$. **It remains to show that any geodesic on $\mathbb{H}$ is a fixed point set of some reflection.**

Let $\gamma$ be a vertical line $x = 0$ on the upper half-plane $\{(x, y) \in \mathbb{R}^2, y > 0\}$ with the metric $\frac{dx^2 + dy^2}{y^2}$. Clearly, $\gamma$ is a fixed point set of a reflection $(x, y) \longrightarrow (-x, y)$. **Since every geodesic is conjugate to $\gamma$, every geodesic is a fixed point set of a reflection.** ■

## Geodesics on the hyperbolic plane

Let $V = \mathbb{R}^3$ be a vector space with quadratic form $q$ of signature (1,2), $\operatorname{Pos} := \{v \in V \mid q(v) > 0\}$, and $\mathbb{P}\operatorname{Pos}$ its projectivisation. Then $\mathbb{P}\operatorname{Pos} = SO^+(1,2)/SO(1)$ **(check this)**, giving $\mathbb{P}\operatorname{Pos} = \mathbb{H}^2$; **this is one of the standard models of a hyperbolic plane.**

**REMARK:** Let $l \subset V$ be **a line,** that is, a 1-dimensional subspace. The property $q(x,x) < 0$ for a non-zero $x \in l$ is written as $q(l,l) < 0$. A line $l$ with $q(l,l) < 0$ is called **negative line** , a line with $q(l,l) > 0$ is called **positive line**.

**PROPOSITION:** Reflections on $\mathbb{P}\operatorname{Pos}$ **are in bijective correspondence with negative lines $l \subset V$.**

*(see the proof on the next slide)*

**REMARK:** Using the equivalence between reflections and geodesics established above, this proposition can be reformulated by saying that **geodesics on $\mathbb{P}\operatorname{Pos}$ are the same as negative lines $l \in \mathbb{P}V$.**

## Geodesics on hyperbolic plane (2)

**PROPOSITION:** Reflections on $\mathbb{P}\operatorname{Pos}$ **are in bijective correspondence with negative lines $l \subset V$.**

**Proof. Step 1:** Consider an isometry $\tau$ of $V$ which fixes $x$ and acts as $v \longrightarrow -v$ on its orthogonal complement $v^\perp$. Since $v^\perp$ has signature (1,1), the set $\mathbb{P}\operatorname{Pos} \cap \mathbb{P}v^\perp$ is 1-dimensional and fixed by $\tau$. We proved that $\tau$ fixes a codimension 1 submanifold in $\mathbb{P}\operatorname{Pos} = \mathbb{H}^2$, hence $\tau$ is a reflection.

It remains to show that **any reflection is obtained this way.**

**Step 2:** Since geodesics are fixed point sets of reflections, and all geodesics are conjugate by isometries, **all reflections are also conjugated by isometries.** Therefore, it suffices to prove that the reflection $x_1, x_2, x_3 \longrightarrow x_1, x_2, -x_3$ is obtained from a negative line $l$. Let $l = (0, 0, \lambda)$. Then $\tau(x_1, x_2, x_3) = -x_1, -x_2, x_3$, and on $\mathbb{P}V$ this operation acts as $x_1, x_2, x_3 \longrightarrow x_1, x_2, -x_3$. ■

**REMARK:** This also implies that **all geodesics in $\mathbb{P}\operatorname{Pos}$ are obtained as intersections $\mathbb{P}\operatorname{Pos} \cap \mathbb{P}W$, where $W \subset V$ is a subspace of signature (1,1).**