Lecture 9: Kobayashi pseudometric

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Upper half-plane as a Riemannian manifold (reminder)

DEFINITION: Let $G = PSL(2, \mathbb{R})$ be a group of orientation-preserving conformal (that is, holomorphic) automorphisms of the upper halfplane $\mathbb{H}^2$. **Poincaré half-plane** is the upper half-plane equipped with a G-invariant metric of constant negative curvature constructed in Lecture 7.

THEOREM: Let (x, y) be the usual coordinates on the upper half-plane $\mathbb{H}^2$, and s a $PSL(2, \mathbb{R})$-invariant metric on $\mathbb{H}^2$. **Then** $s = const \frac{dx^2 + dy^2}{y^2}$.

DEFINITION: Minimising geodesic in a Riemannian manifold is a piecewise smooth path connecting x to y such that its length is equal to the geodesic distance. **Geodesic** is a piecewise smooth path γ such that for any $x \in \gamma$ there exists a neighbourhood of x in γ which is a minimising geodesic.

THEOREM: Geodesics on a Poincaré half-plane are vertical half-lines and their images under the action of $PSL(2, \mathbb{R})$.

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Poincaré metric on a disk (reminder)

DEFINITION: Poincaré metric on the unit disk $\Delta \subset \mathbb{C}$ is an Aut(Δ)-invariant metric **(it is unique up to a constant multiplier; prove it)**.

DEFINITION: Let $f : \ M \longrightarrow M_1$ be a map of metric spaces. Then f is called C-**Lipschitz** if $d(x, y) \geqslant Cd(f(x), f(y))$. A map is called **Lipschitz** if it is C-Lipschitz for some $C > 0$.

THEOREM: (Schwartz-Pick lemma)
Any holomorphic map $\varphi : \ \Delta \longrightarrow \Delta$ from a unit disk to itself is 1-Lipschitz with respect to Poicaré metric.

Proof. Step 1: We need to prove that for each $x \in \Delta$ the norm of the differential, taken with respect to the Poincaré metric, satisfies $|D\varphi_x|_P \leqslant 1$. Since the automorphism group acts on Δ transitively, **it suffices to prove that $|D\varphi_x| \leqslant 1$ when $x = 0$ and $\varphi(x) = 0$.**

Step 2: This is Schwartz lemma. ∎

Kobayashi pseudometric

DEFINITION: Pseudometric on M is a function $d: M \times M \longrightarrow \mathbb{R}^{\geqslant 0}$ which is symmetric: $d(x, y) = d(y, x)$ and satisfies the triangle inequality $d(x, y) + d(y, z) \geqslant d(x, z)$.

REMARK: Let $\mathfrak{D}$ be a set of pseudometrics. **Then** $d_{\max}(x, y) := \sup_{d \in \mathfrak{D}} d(x, y)$ **is also a pseudometric (prove this as an exercise).**

REMARK: This is false for inf **(construct a counterexample)**.

DEFINITION: The **Kobayashi pseudometric** on a complex manifold M is $d_{\max}$ for the set $\mathfrak{D}$ of all pseudometrics such that any holomorphic map from the Poincaré disk to M is distance-decreasing.

EXERCISE: Prove that the distance between points x, y in Kobayashi pseudometric is infimum of the Poincaré distance over all sets of Poincaré disks connecting x to y.

CLAIM: Any holomorphic map $X \xrightarrow{\varphi} Y$ is 1-Lipschitz with respect to the Kobayashi pseudometric.

Proof: If $x \in X$ is connected to x' by a sequence of Poincare disks $\Delta_1, ..., \Delta_n$, then $\varphi(x)$ is connected to $\varphi(x')$ by $\varphi(\Delta_1), ..., \varphi(\Delta_n)$. ∎

REMARK: This implies that **the Kobayashi pseudometric on $\mathbb{C}$ vanishes.** Indeed, homothety must be an isometry, which is impossible unless the metric vanishes **(prove this)**.

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Kobayashi hyperbolic manifolds

COROLLARY: Let $B \subset \mathbb{C}^n$ be a unit ball, and $x, y \in B$ points with coordinates $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$. Since x_i, y_i belongs to Δ, it makes sense to compute the Poincare distance $d_P(x_i, y_i)$. **Then** $d_K(x, y) \geqslant \max_i d_P(x_i, y_i)$.

Proof: Each of projection maps $\Pi_i : B \longrightarrow \Delta$ is 1-Lipshitz, hence the Kobayashi metric is bounded from below by each of $d_P(x_i, y_i)$. ∎

DEFINITION: A variety is called **Kobayashi hyperbolic** if the Kobayashi pseudometric d_K is non-degenerate.

DEFINITION: A **domain** in $\mathbb{C}^n$ is an open subset. A **bounded domain** is an open subset contained in a ball.

COROLLARY: Any bounded domain Ω in $\mathbb{C}^n$ is Kobayashi hyperbolic.

Proof: Without restricting generality, we may assume that $\Omega \subset B$ where B is an open ball. Then the Kobayashi distance in Ω is $\geqslant$ that in B. However, the Kobayashi distance in B is bounded by the metric $d(x, y) := \max_i d_P(x_i, y_i)$ as follows from above. ∎

Caratheodory metric

DEFINITION: Let $x, y \in M$ be points on a complex manifold. Define **Caratheodory pseudometric** as $d_C(x, y) = \sup\{d_P(f(x), f(y))\}$, where the supremum is taken over all holomorphic map $f : M \longrightarrow \Delta$, and d_P is Poincare metric on the disk Δ.

REMARK: Usually the term "Kobayashi/Caratheodory pseudometric" is abbreviated to "Kobayashi/Caratheodory metric", **even when it is not a metric.**

REMARK: Caratheodory pseudometric **satisfies the triangle inequality** because a supremum of pseudometrics satisfies triangle inequality.

EXERCISE: Prove that **Caratheodory pseudometric is bounded by the Kobayashi pseudometric:** $d_K \geqslant d_C$.

REMARK: Clearly, $d_C \neq 0$ on any bounded domain.

Complex hyperbolic space

DEFINITION: Let $V = \mathbb{C}^{n+1}$ be a complex vector space equipped with a Hermitian metric h of signature $(1, n)$, and $\mathbb{H}^n_{\mathbb{C}} \subset \mathbb{P}V$ projectivization of the set of positive vectors $\{x \in V \parallel h(x, \bar{x}) > 0\}$. Then $\mathbb{H}^n_{\mathbb{C}}$ is equipped with a homogeneous action of $U(1, n)$. The same argument as used for space forms implies that $\mathbb{H}^n_{\mathbb{C}}$ admits a $U(1, n)$-invariant Hermitian metric, which is unique up to a constant multiplier. This Hermitian complex manifold is called **complex hyperbolic space**.

REMARK: For $n > 1$ **it is not isometric to the real hyperbolic spaces defined earlier.**

REMARK: As a complex manifold $\mathbb{H}^n_{\mathbb{C}}$ **is isomorphic to an open ball in $\mathbb{C}^n$ (prove it!)**

REMARK: The Kobayashi metric and the Caratheodory metric on $\mathbb{H}^n_{\mathbb{C}}$ are $U(1, n)$-invariant, because $U(1, n)$ acts holomorphically, hence proportional to the hyperbolic metric, which is also called **Bergman metric** on an open ball.

EXERCISE: Prove that **Kobayashi metric on a ball in $\mathbb{C}^n$ is equal to the Caratheodory metric.**