Lecture 10: Arzelà-Ascoli theorem and Montel theorem

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Limits maps and completions

DEFINITION: A map $f: M \longrightarrow N$ of metric spaces is **C-Lipschitz** if $d(f(x), f(y)) \leqslant Cd(x, y)$ for all $x, y \in M$.

Claim 0: Let $M' \subset M$ be a dense subset. **Then any C-Lipschitz map $f: M' \longrightarrow N$ can be extended to a C-Lipschitz map $\tilde{f}: M \longrightarrow N$.**

Proof: Clearly, f maps Cauchy sequences in M' to Cauchy sequences, and equivalent Cauchy sequences to equivalent Cauchy sequences. Also, M can be identified with the metric completion of M', that is, with the set of equivalences classes of Cauchy sequences. This defines the map $\tilde{f}$. It is C-Lipschitz because for any Cauchy sequences $\{x_i\}$, $\{y_i\} \subset M$ converging to x, y, the distance in the metric completion is defined as $d(x, y) = \lim d(x_i, y_i)$. ■

DEFINITION: A sequence of maps $f_i: M \longrightarrow N$ between metric spaces **uniformly converges** (or **converges uniformly on compacts**) to $f: M \longrightarrow N$ if for any compact $K \subset M$, we have $\lim\limits_{i\to\infty} \sup_{x \in K} d(f_i(x), f(x)) = 0$.

Uniform convergence for Lipschitz maps

Claim 1: Suppose that a sequence $f_i: M \longrightarrow N$ of 1-Lipschitz maps converges to f pointwise in a countable dense subset $M' \subset M$. **Then f_i converges to f uniformly on compacts.**

Proof. Step 1: Let $K \subset M$ be a compact set, and $N_\varepsilon \subset M'$ a finite subset such that K is a union of ε-balls centered in N_ε (such N_ε is called **an ε-net**). **Then there exists N such that $\sup_{x\in N_\varepsilon} d(f_{N+i}(x), f(x)) < \varepsilon$ for all $i \geqslant 0$.**

Step 2: Since f_i are 1-Lipschitz, this implies that

$$\sup_{y\in K} d(f_{N+i}(y), f(y)) \leqslant$$
$$\leqslant d(f_{N+i}(x), f(x)) + (d(f_{N+i}(x), f_{N+i}(y)) + d(f(x), f(y)) \leqslant 3\varepsilon,$$

where $x \in N_\varepsilon$ is chosen in such a way that $d(x,y) < \varepsilon$ for all $y \in K$. **In this inequality, each of the 3 terms in the middle is $\leqslant \varepsilon$.** The first term is $\leqslant \varepsilon$ by Step 1, the second because f_{N+i} is 1-Lipschitz, and the third by Claim 0.
■

EXERCISE: Prove that the limit f is also 1-Lipschitz.

REMARK: This proof works when M is a pseudo-metric space, as long as N is a metric space.

Uniform convergence for Lipschitz maps (2)

COROLLARY 1: The space of Lipschitz maps **is closed in the topology of pointwise convergence.** Moreover, **pointwise convergence of Lipschitz maps implies uniform convergence on compacts.**

Proof: The same inequality as above **(do this as an exercise).** ■

DEFINITION: Let M, N be metric spaces. A subset $B \subset N$ is **bounded** if it is contained in a ball of finite radius. A family $\{f_\alpha\}$ of maps $f_\alpha : M \longrightarrow N$ is called **uniformly bounded on compacts** if for any compact subset $K \subset M$, there is a bounded subset $C_K \subset N$ such that $f_\alpha(K) \subset C_K$ for any element f_α of the family.

Arzelà-Ascoli theorem for Lipschitz maps

THEOREM: (Arzelà-Ascoli for Lipschitz maps)
Let M be a metric space and $\mathcal{F} := \{f_\alpha\}$ a set of 1-Lipschitz maps $f_\alpha : M \longrightarrow \mathbb{C}$, uniformly bounded on compacts. Assume that M has countable base of open sets and can be obtained as a countable union of compact subsets. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to a function $f : M \longrightarrow \mathbb{C}$ uniformly on compacts.**

REMARK: The limit f is clearly also 1-Lipschitz. Indeed, a uniform convergent family is pointwise convergent, and a pointwise limit of 1-Lipschitz maps is also 1-Lipschitz (Corollary 1).

Proof. Step 1: Suppose we can prove Arzelà-Ascoli when M is compact. Then we can choose a sequence of compact subsets $K_i \subset M$, find subsequences in $\mathcal{F}$ converging on each K_i, and use the diagonal method to find a subsequence converging on all K_i. Therefore, **we can assume that M is compact.**

Step 2. By definition of pointwise convergence, for any finite set $S \subset M$, there exists a subsequence f_i of $\mathcal{F}$ which converges to $f \in \operatorname{Map}(S, N)$ in S. Using the diagonal method, we choose a subsequence f_i of $\mathcal{F}$ which converges to $f \in \operatorname{Map}(M', N)$ pointwise in a dense countable set $M' \subset M$. **Then f_i converges to f uniformly by Claim 1.** ■

Arzelà-Ascoli theorem for Lipschitz maps (a second proof)

THEOREM: (Arzelà-Ascoli for Lipschitz maps)
Let $\mathcal{F} := \{f_\alpha\}$ be an infinite set of 1-Lipschitz maps $f_\alpha : M \longrightarrow \mathbb{C}$, uniformly bounded on compacts. Assume that M has countable base of open sets and can be obtained as a countable union of compact subsets. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : M \longrightarrow \mathbb{C}$ uniformly on compacts.**

Proof. Step 1: Using the diagonal argument, **we can assume that M is compact, and all maps $f_\alpha : M \longrightarrow \mathbb{C}$ map M into a compact subset $N \subset \mathbb{C}$.** It remains to show that the space of Lipschitz maps from M to N is compact with topology of uniform convergence.

Step 2. The space of maps to a compact is compact in topology of pointwise convergence **(Tychonoff theorem)**. However, **on Lipschitz maps, pointwise convergence implies uniform convergence** (Corollary 1). ■

CLAIM: Arzelà-Ascoli is also true when M is a pseudo-metric space.

Proof: Consider the metric quotient space $M_1 := M/\sim$, where $x \sim y \Leftrightarrow d(x,y) = 0$, and the natural projection $\pi : M \longrightarrow M_1$. Then any 1-Lipschitz function is constant on the sets $\pi^{-1}(x)$, for all $x \in M$, and defines a Lipschitz function on M_1. ■

Giulio Ascoli, Cesare Arzelà

Giulio Ascoli,
1843-1896

Cesare Arzelà
1847-1912

...The notion of equicontinuity was introduced in the late 19th century by the Italian mathematicians Cesare Arzelà and Giulio Ascoli. A weak form of the theorem was proven by Ascoli (1883-1884), who established the sufficient condition for compactness, and by Arzelà (1895), who established the necessary condition and gave the first clear presentation of the result.

In the last moment I found that Wikipedia has a picture of the medic Giulio Ascoli (1870-1916) in place of the mathematician! It seems that the picture of the mathematician Giulio Ascoli is not available online... Politecnico di Milano might still have a plague in his name affixed to it in 1901 (not likely, because Ascoli was Jewish).

Kobayashi pseudometric (reminder)

DEFINITION: Pseudometric on M is a function $d: M\times M \longrightarrow \mathbb{R}^{\geqslant 0}$ which is symmetric: $d(x,y)=d(y,x)$ and satisfies the triangle inequality $d(x,y)+d(y,z)\geqslant d(x,z)$.
REMARK: Let $\mathfrak{D}$ be a set of pseudometrics. **Then $d_{\max}(x,y) := \sup_{d\in\mathfrak{D}} d(x,y)$ is also a pseudometric (prove this as an exercise).**
REMARK: This is false for inf **(construct a counterexample)**.

DEFINITION: The **Kobayashi pseudometric** on a complex manifold M is $d_{\max}$ for the set $\mathfrak{D}$ of all pseudometrics such that any holomorphic map from the Poincaré disk to M is distance-decreasing.
EXERCISE: Prove that **the distance between points x, y in Kobayashi pseudometric is infimum of the Poincaré distance over all sets of Poincaré disks connecting x to y.**

CLAIM: Any holomorphic map $X \stackrel{\varphi}{\longrightarrow} Y$ is 1-Lipschitz with respect to the Kobayashi pseudometric.
Proof: If $x\in X$ is connected to x' by a sequence of Poincare disks $\Delta_1, ..., \Delta_n$, then $\varphi(x)$ is connected to $\varphi(x')$ by $\varphi(\Delta_1), ..., \varphi(\Delta_n)$. ■

REMARK: This implies that **the Kobayashi pseudometric on $\mathbb{C}$ vanishes.** Indeed, homothety must be an isometry, which is impossible unless the metric vanishes **(prove this)**.

Kobayashi hyperbolic manifolds (reminder)

COROLLARY: Let $B \subset \mathbb{C}^n$ be a unit ball, and $x, y \in B$ points with coordinates $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$. Since x_i, y_i belongs to Δ, it makes sense to compute the Poincare distance $d_P(x_i, y_i)$. **Then $d_K(x, y) \geqslant \max_i d_P(x_i, y_i)$.**

Proof: Each of projection maps $\Pi_i : B \longrightarrow \Delta$ is 1-Lipshitz, hence the Kobayashi metric is bounded from below by each of $d_P(x_i, y_i)$. ■

DEFINITION: A variety is called **Kobayashi hyperbolic** if the Kobayashi pseudometric d_K is non-degenerate.

DEFINITION: A **domain** in $\mathbb{C}^n$ is an open subset. A **bounded domain** is an open subset contained in a ball.

COROLLARY: Any bounded domain Ω in $\mathbb{C}^n$ is Kobayashi hyperbolic.

Proof: Without restricting generality, we may assume that $\Omega \subset B$ where B is an open ball. Then the Kobayashi distance in Ω is $\geqslant$ that in B. However, the Kobayashi distance in B is bounded by the metric $d(x, y) := \max_i d_P(x_i, y_i)$ as follows from above. ■

Normal families of holomorphic functions

DEFINITION: Let M be a complex manifold. A family $\mathcal{F} := \{f_\alpha\}$ of holomorphic functions $f_\alpha : M \longrightarrow \mathbb{C}$ is called **normal family** if $\mathcal{F}$ is uniformly bounded on compact subsets.

THEOREM: (Montel's theorem)
Let M be a complex manifold with countable base, and $\mathcal{F}$ a normal, infinite family of holomorphic functions. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : M \longrightarrow \mathbb{C}$ uniformly on compacts,** and f is holomorphic.

Proof. Step 1: As in the first step of Arzelà-Ascoli, it suffices to prove Montel's theorem on a subset of M where $\mathcal{F}$ is bounded. Therefore, **we may assume that all f_α map M into a disk Δ.**

Step 2: All f_α are 1-Lipschitz with respect to Kobayashi metric. Therefore, **Arzelà-Ascoli theorem can be applied, giving a uniform limit $f = \lim f_i$.**

Step 3: A uniform limit of holomorphic functions is holomorphic by Cauchy formula. ■

REMARK: The sequence $f = \lim f_i$ converges uniformly with all derivatives, again by Cauchy formula.