Lecture 11: Riemann mapping theorem

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Uniform convergence (reminder)

DEFINITION: A sequence of maps $f_i : M \longrightarrow N$ between metric spaces **uniformly converges** (or **converges uniformly on compacts**) to $f : M \longrightarrow N$ if for any compact $K \subset M$, we have $\lim_{i\to\infty} \sup_{x\in K} d(f_i(x), f(x)) = 0$.

Claim 1: Suppose that a sequence $f_i : M \longrightarrow N$ of 1-Lipschitz maps converges to f pointwise in a countable dense subset $M' \subset M$. **Then f_i converges to f uniformly on compacts.**

COROLLARY 1: The space of Lipschitz maps **is closed in the topology of pointwise convergence.** Moreover, **pointwise convergence of Lipschitz maps implies uniform convergence on compacts.**

Arzelà-Ascoli theorem for Lipschitz maps (reminder)

DEFINITION: Let M, N be metric spaces. A subset $B \subset N$ is **bounded** if it is contained in a ball of finite radius. A family $\{f_\alpha\}$ of maps $f_\alpha : M \longrightarrow N$ is called **uniformly bounded on compacts** if for any compact subset $K \subset M$, there is a bounded subset $C_K \subset N$ such that $f_\alpha(K) \subset C_K$ for any element f_α of the family.

THEOREM: (Arzelà-Ascoli for Lipschitz maps)
Let M be a metric space and $\mathcal{F} := \{f_\alpha\}$ a set of 1-Lipschitz maps $f_\alpha : M \longrightarrow \mathbb{C}$, uniformly bounded on compacts. Assume that M has countable base of open sets and can be obtained as a countable union of compact subsets. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to a function $f : M \longrightarrow \mathbb{C}$ uniformly on compacts.**

REMARK: The limit f is also 1-Lipschitz. Indeed, a uniform convergent family is pointwise convergent, and a pointwise limit of 1-Lipschitz maps is also 1-Lipschitz (Corollary 1).

Normal families of holomorphic functions (reminder)

DEFINITION: Let M be a complex manifold. A family $\mathcal{F} := \{f_\alpha\}$ of holomorphic functions $f_\alpha : M \longrightarrow \mathbb{C}$ is called **normal family** if $\mathcal{F}$ is uniformly bounded on compact subsets.

THEOREM: (Montel's theorem)
Let M be a complex manifold with countable base, and $\mathcal{F}$ a normal, infinite family of holomorphic functions. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : M \longrightarrow \mathbb{C}$ uniformly on compacts,** and f is holomorphic.

Proof. Step 1: As in the first step of Arzelà-Ascoli, it suffices to prove Montel's theorem on a subset of M where $\mathcal{F}$ is bounded. Therefore, **we may assume that all f_α map M into a disk Δ.**

Step 2: All f_α are 1-Lipschitz with respect to Kobayashi metric. Therefore, **Arzelà-Ascoli theorem can be applied, giving a uniform limit $f = \lim f_i$.**

Step 3: A uniform limit of holomorphic functions is holomorphic by Cauchy formula. ■

REMARK: The sequence $f = \lim f_i$ converges uniformly with all derivatives, again by Cauchy formula.

Riemann mapping theorem

THEOREM: Let $\Omega \subset \Delta$ be a simply connected, bounded domain. **Then Ω is biholomorphic to Δ.**

Idea of a proof: We consider the Kobayashi metric on Ω and Δ, and let $\mathcal{F}$ be the set of all injective holomorphic maps $\Omega \longrightarrow \Delta$. Consider $x \in \Omega$, and let f be a map with $|df_x|$ maximal in the sense of Kobayashi metric in the closure of $\mathcal{F}$. **Such f exists by Montel's theorem. We prove that f is a bijective isometry**, and hence biholomorphic.

Functions which take distinct values on a boundary of a disk

Lemma 1: Let u, v be non-equal holomorphic functions on a disk Δ, and $S_r \subset \Delta$ a circle of radius zero around 0. **Then for all $r \in]0,1]$, except a countable set, $u(z) \neq v(z)$ for all $z \in S_r$.**

Proof: The set of zeros of $u(z) - v(z)$ is discrete in Δ, hence countable. Therefore, for all r except countably many, S_r does not contains zeros of $u(z) - v(z)$. ■

Lemma 2: Let $\mathcal{R}$ be the set of all pairs of distinct, non-constant holomorphic functions $g, h: \Delta \longrightarrow \mathbb{C}$ continuously extended to the boundary such that $h(x) = g(x)$ for some $x \in \Delta$, but $h(x) \neq g(x)$ everywhere on the boundary. **Then $\mathcal{R}$ is open in uniform topology.**

Proof. Step 1: Consider the function $\frac{(h-g)'}{h-g}$ on Δ. **This function has a simple pole in all the points where $h = g$.** Moreover, $n_{h,g} := \frac{1}{\pi\sqrt{-1}} \int_{\partial\Delta} dz$ is equal to the number of points $x \in \Delta$ such that $h(x) = g(x)$ (taken with multiplicities, which are always positive integers).

Step 2: Since the integral is continuous in uniform topology, **this number is locally constant on the space of pairs such $h, g: \Delta \longrightarrow \mathbb{C}$.** Therefore, the set $\mathcal{R}$ of all h, g with $n_{h,g} \neq 0$ is open. ■

The set of injective holomorphic maps is closed

PROPOSITION: Let $\mathcal{H}$ be the set of holomorphic maps $f: \Omega_1 \longrightarrow \Omega_2$ between Riemann surfaces, equipped with uniform topology, and $\mathcal{H}_0$ its subset consisting of injective maps and constant maps. **Then $\mathcal{H}_0$ is closed in $\mathcal{H}$.**

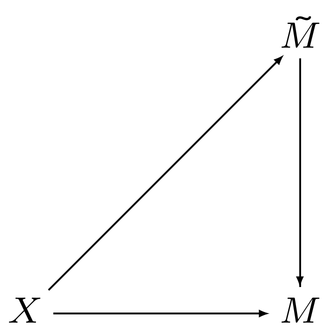
Proof. Step 1: We will prove that **the set of non-injective maps is open in uniform topology.**

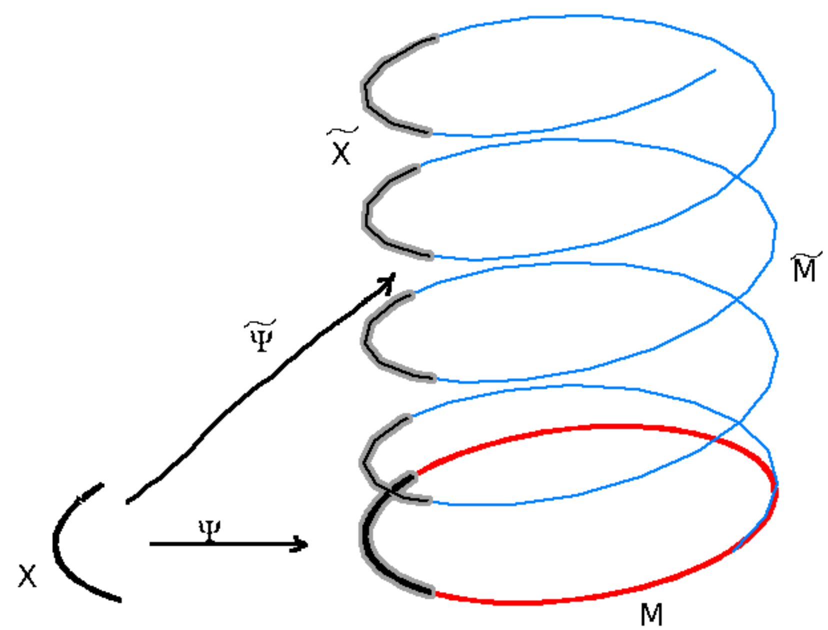
Step 2: Let $f: \Omega_1 \longrightarrow \Omega_2$ be a non-injective mapp Then $f(a) = f(b)$ for some $a \neq b$ in Ω_1. Choose open disks A and $B \subset \Omega_1$ containing a and b. Using Lemma 1, we may shrink A and B, and identify A and B with Δ in such a way that a and b correspond to 0, and the functions g and h obtained by restricting f to $\partial A = \partial B$ satisfy $g - h \neq 0$ everywhere on $\partial A = \partial B$. **By Lemma 2, for any f_1 in a sufficiently small uniform neighbourhood of f, the function $g_1 := f_1|_A$ and $h_1 := f_1|_B = A$ also satisfy $h_1(z) = g_1(z)$ for some $z \in B = A$.** Therefore, f_1 is also non-injective. ■

Coverings (reminder)

DEFINITION: A topological space X is **locally path connected** if for each $x \in X$ and each neighbourhood $U \ni x$, there exists a smaller neighbourhood $W \ni x$ which is path connected.

THEOREM: (homotopy lifting principle)
Let X be a simply connected, locally path connected topological space, and $\tilde{M} \longrightarrow M$ a covering map. Then for each continuous map $X \longrightarrow M$, there exists a lifting $X \longrightarrow \tilde{M}$ making the following diagram commutative.

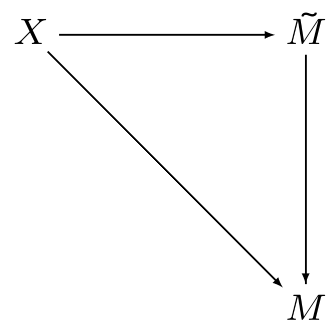




Homotopy lifting principle (reminder)

EXAMPLE: The map $x \longrightarrow x^2$ is a covering from $\mathbb{C}^* := \mathbb{C}\backslash 0$ to itself **(prove it)**.

THEOREM: (homotopy lifting principle)
Let X be a simply connected, locally path connected topological space, and $\tilde M \longrightarrow M$ a covering map. Then for each continuous map $X \longrightarrow M$, there exists a lifting $X \longrightarrow \tilde M$ making the following diagram commutative.



COROLLARY: Let $\varphi : \Omega \longrightarrow \mathbb{C}^*$ be a holomorphic map from a simply connected domain Ω. **Then there exists a holomorphic map $\varphi_1 : \Omega \longrightarrow \mathbb{C}^*$ such that for all $z \in \Delta$, $\varphi(z) = \varphi_1(z)^2$.**

Proof: We apply homotopy lifting principle to $X = \Omega$, $M = \tilde M = \mathbb{C}^*$, and $\tilde M \longrightarrow M$ mapping x to x^2. ■

REMARK: We denote $\varphi_1(z)$ by $\sqrt{\varphi(z)}$, for obvious reasons.

Poincaré metric and the map $x \longrightarrow x^2$

CLAIM: Consider a non-bijective holomorphic map $\varphi: \Delta \longrightarrow \Delta$ from Poincare disk to itself. **Then $|d\varphi| < 1$ at each point,** where $d\varphi$ is a norm of an operator $d\varphi: T_x\Delta \longrightarrow T_{\varphi(x)}\Delta$ taken with respect to the Poincare metric.

Proof: Let $\varphi: \Delta \longrightarrow \Delta$ be a holomorphic map which satisfies $|d\varphi| = 1$ at $x \in \Delta$. Replacing φ by $\gamma_1 \circ \varphi \circ \gamma_2$ if necessary, where γ_i are biholomorphic isometries of Δ, we may assume that $x = 0$ and $\varphi(x) = 0$. By Schwartz lemma, for such φ, relation $|d\varphi(0)| = 1$ implies that φ is a linear biholomorphic map. ■

REMARK: We will apply this claim only to the function $x \xrightarrow{\varphi} x^2$. However, **even for this function it takes some work,** because an explicit proof needs and explicit form of Poincaré metric on a disk, which we did not have.

Poincaré metric and $\sqrt{\varphi}$

Corollary 2: Let $\varphi: \Delta \longrightarrow \Delta\backslash 0$ be a holomorphic function, and $\sqrt{\varphi}$ a holomorphic function defined above. Let $|d\varphi|(x)$ denote the norm of the operator $d\varphi$ at $x \in \Delta$ computed with respect to the Poincare metric on Δ. **Then $|d\varphi|(x) < |d\sqrt{\varphi}|(x)$ for any $x \in \Delta$.**

Proof: Let $\psi(x) = x^2$. By the claim above, $|d\psi|(x) < 1$ for all $x \in \Delta$ (here **the norm is taken with respect to Poincaré metric**). Using the chain rule, we obtain that $d\varphi = d\psi \circ d\sqrt{\varphi}$. which gives $|d\varphi|(x) = |d\psi|(\sqrt{\varphi}(x))|d\sqrt{\varphi}|(x)$, hence

$$|d\sqrt{\varphi}|(x) = \frac{|d\varphi|(x)|}{|d\psi|(\sqrt{\varphi}(x))} > |d\varphi|(x).$$

■

Riemann mapping theorem

THEOREM: Let $\Omega \subset \Delta$ be a simply connected domain. **Then Ω is biholomorphic to Δ.**

Proof. Step 1: Consider the Kobayashi metric on Ω and Δ, and let $\mathcal{F}$ be the set of all injective holomorphic maps $\Omega \longrightarrow \Delta$. Consider $x \in \Omega$, and let f be a map with $|df|(x)$ maximal in the sense of Kobayashi metric. **Such f exists by Montel's theorem.** Since f lies in the closure of $\mathcal{F}$, and the set of injective maps is closed, **f is injective.**

Step 2: It remains to show that f is surjective. Suppose it is not surjective: $z \notin f(\Omega)$. Taking a composition of f and an isometry of the Poincare disk does not affect $|df|(x)$, hence we may assume that $z = 0$. **Then the function $\sqrt{f}$ is a well defined holomorphic map from Ω to Δ.** By Corollary 2, $|d\sqrt{f}|(x) > |df|(x)$, which is impossible, because it $|df|(x)$ is maximal. ■

Fatou and Julia sets

DEFINITION: Let X, Y be complex varieties, and $\mathcal{F}$ a family of holomorphic maps $f_\alpha: X \longrightarrow Y$. Recall that $\mathcal{F}$ is **a normal family** if any sequence $\{f_i\}$ in $\mathcal{F}$ has a subsequence which converges (uniformly on compacts) to a holomorphic map.

DEFINITION: Let $f: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a rational map, and $\{f^i\} = \{f, f \circ f, f \circ f \circ f, ...\}$ the set of all iterations of f. **Fatou set of f** is the set of all points $x \in \mathbb{C}$ such that for some neighbourhood $U \ni x$, the restriction $\{f^i|_U\}$ is a normal family, with all f^i except finitely many taking values in a disk $B \subset \mathbb{C}P^1$, and **Julia set** is a complement to Fatou set.

EXAMPLE: For the map $f(x) = x^2$, Julia set is the unit circle, and the Fatou set is its complement **(prove it)**.

Attractor points

DEFINITION: Attractor point z is a fixed point of f such that $|df|(z) < 1$; the **attractor basin** for z is the set of all $x \in \mathbb{C}$ such that $\lim_i f^i(x) = z$.

CLAIM: For any fixed point z, **its attractor basin belongs to the Fatou set.**

Proof: Indeed, since $\lim_i f^i(x) = z$ for any point in attractor basin U, $\{f^i\}$ is a normal family on U **(pointwise convergence is equivalent to uniform convergence for bounded holomorphic functions by Arzela-Ascoli theorem).** ■

Clearly, z **is an attracting fixed point of** $f(t)$ **if** $f(z) = z$ **and** $|f'(z)| < 1$**.**

Newton iteration method

DEFINITION: Newton iteration for solving the polynomial equation $g(z) = 0$: a solution is obtained as a limit $\lim_i f^i(z)$, where $f(z) = z - \frac{g(z)}{g'(z)}$.

CLAIM: The solutions of $g(z) = 0$ are attracting fixed points of f.

Proof. Step 1: Let z be a solution of $g(z) = 0$, and m the multiplicity of zero of g in z. Since $g'(z)$ haz zero of multiplicity $m-1$, one has $\frac{g(z)}{g'(z)} = 0$, the function $\frac{g(z)}{g'(z)}$ is holomorphic in a neighbourhood of z by Riemann removable singularity theorem, and $f(z) = z$.

Step 2: To simplify the formulas, assume that $z = 0$. Since $g(0) = 0$ and g has a zero of multiplicity m, the Taylor decomposition for g in 0 takes form

$$g(z) = a_m z^m + a_{m+1} z^{m+1} + ...,$$

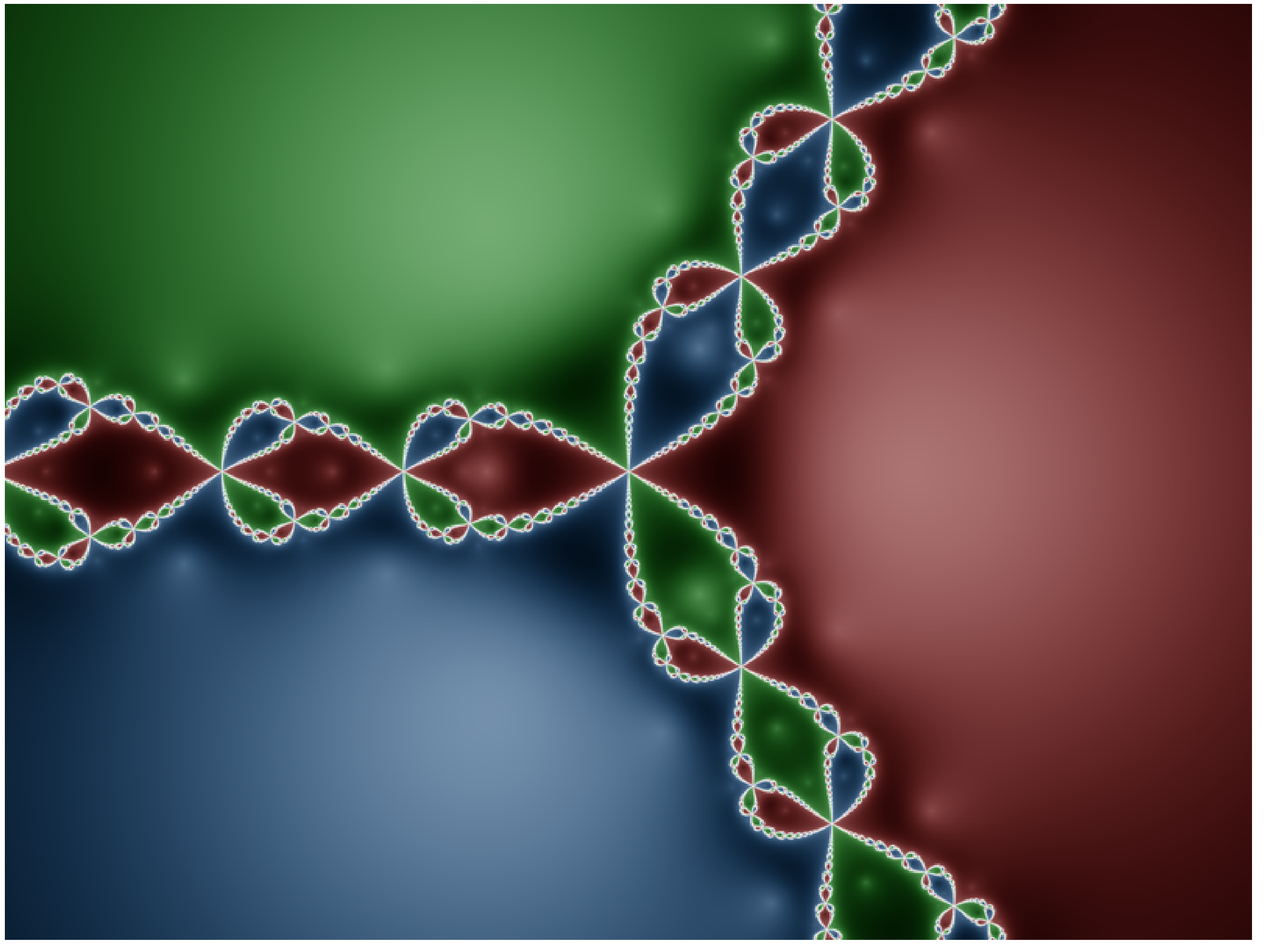
where $a_m \neq 0$. Then

$$g'(z) = m a_m z^{m-1} + (m+1) a_{m+1} z^m + ...$$

Let $u(z) = a_m + a_{m+1} z + a_{m+2} z^2 + ...$ and $v(z) = m a_m + (m+1) a_{m+1} z + (m+2) a_{m+2} z^2 + ...$ Since $\sum |a_i|$ converges, these series converge in an appropriate neighbourhood of zero. Clearly, $g(z) = z^m u(z)$ and $g'(z) = z^{m-1} u(z)$, which gives $\frac{g(z)}{g'(z)} = z \frac{u(z)}{v(z)}$, and $\frac{d}{dz} \frac{g(z)}{g'(z)}|_{z=0} = \frac{u(0)}{v(0)} = \frac{1}{m}$. This gives $f'(0) = 1 - \frac{1}{m}$. ■

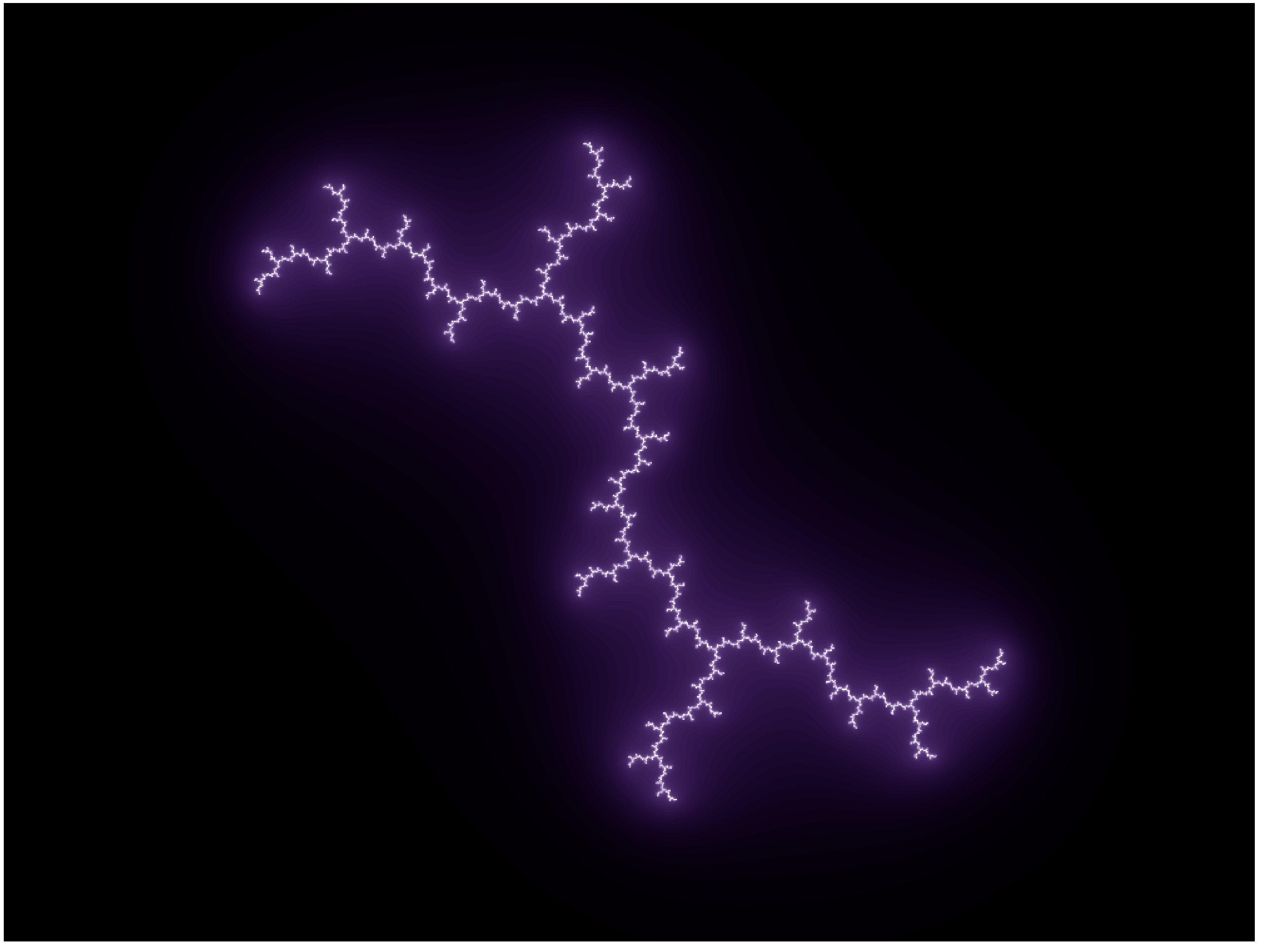
Fatou and Julia sets for $f(z) = \frac{1+2z^3}{3z^2}$

We apply the Newton iteration method to $g(z) = z^3 - 1$.



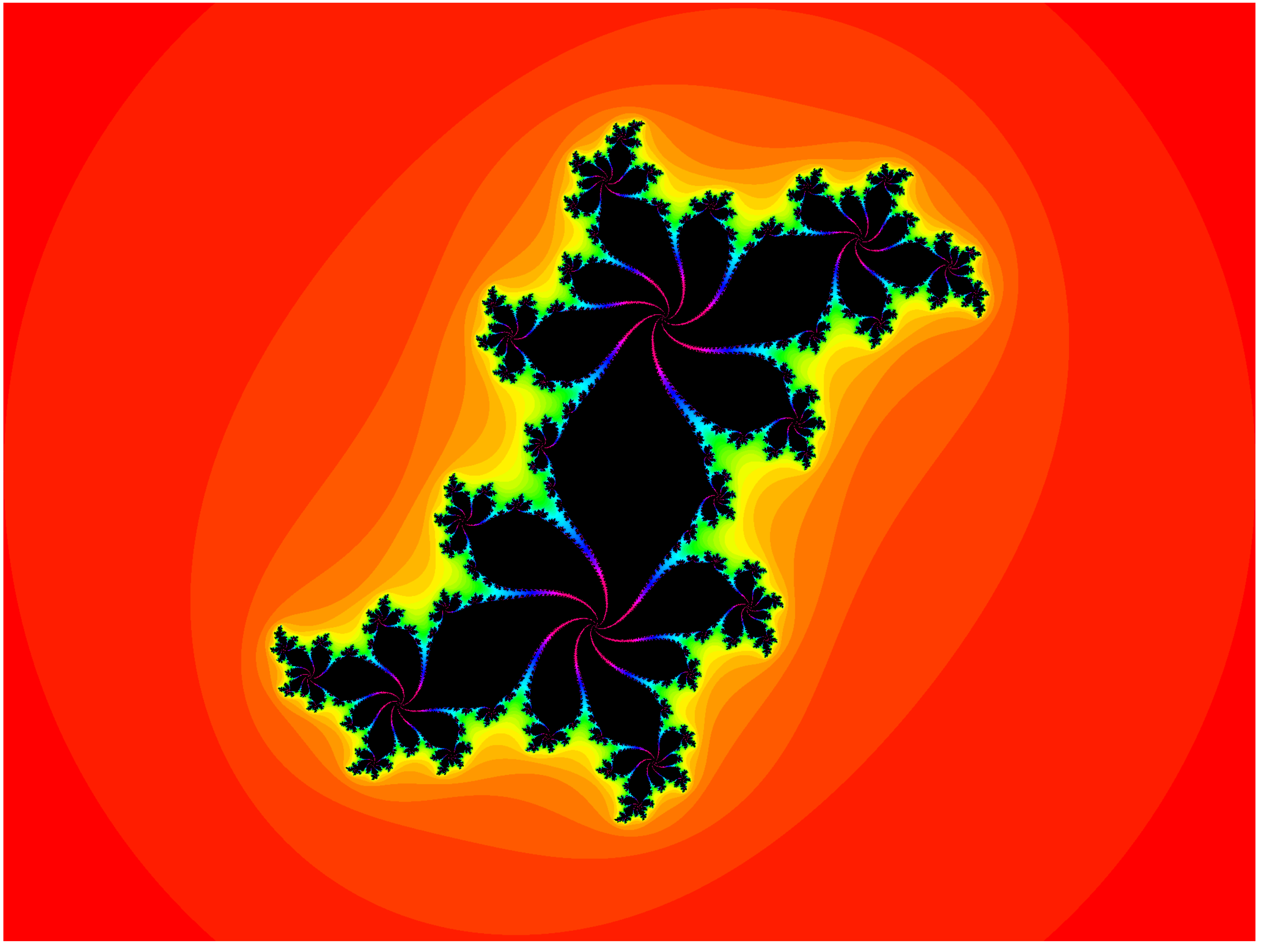
Julia set (in white) for the map $f(z) = \frac{1+2z^3}{3z^2} = z - \frac{g(z)}{g'(z)}$. Attractor basins for three roots of $g(z) = z^3 - 1$ are colored in red, green, blue.

Julia set for $f(z) = z^2 - \sqrt{-1}$

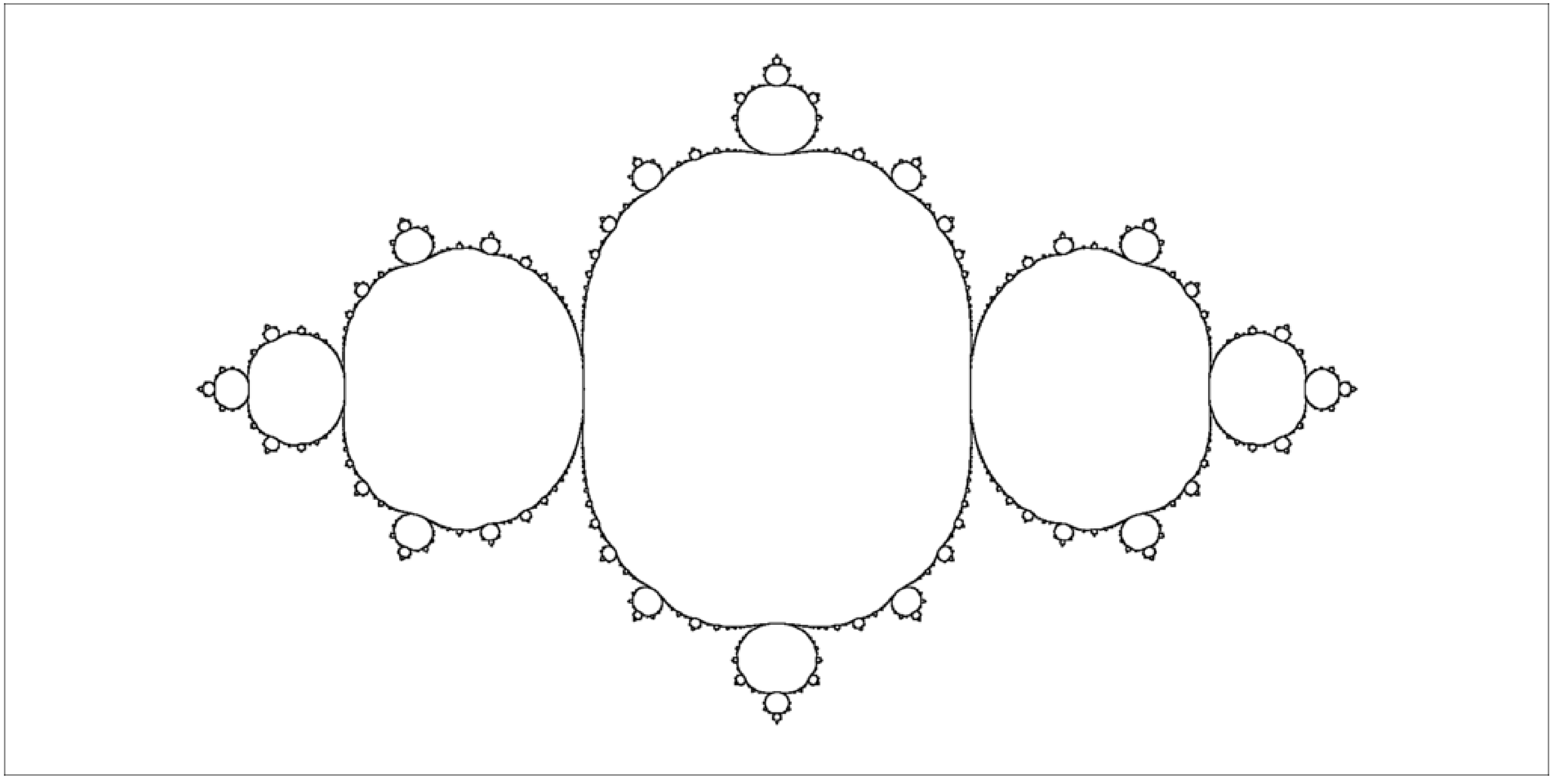


Julia set for $f(z) = z^2 - \sqrt{-1}$ is called **dendrite**.

Julia set for $f(z) = z^2 + 0.12 + 0.6\sqrt{-1}$



San Marco fractal

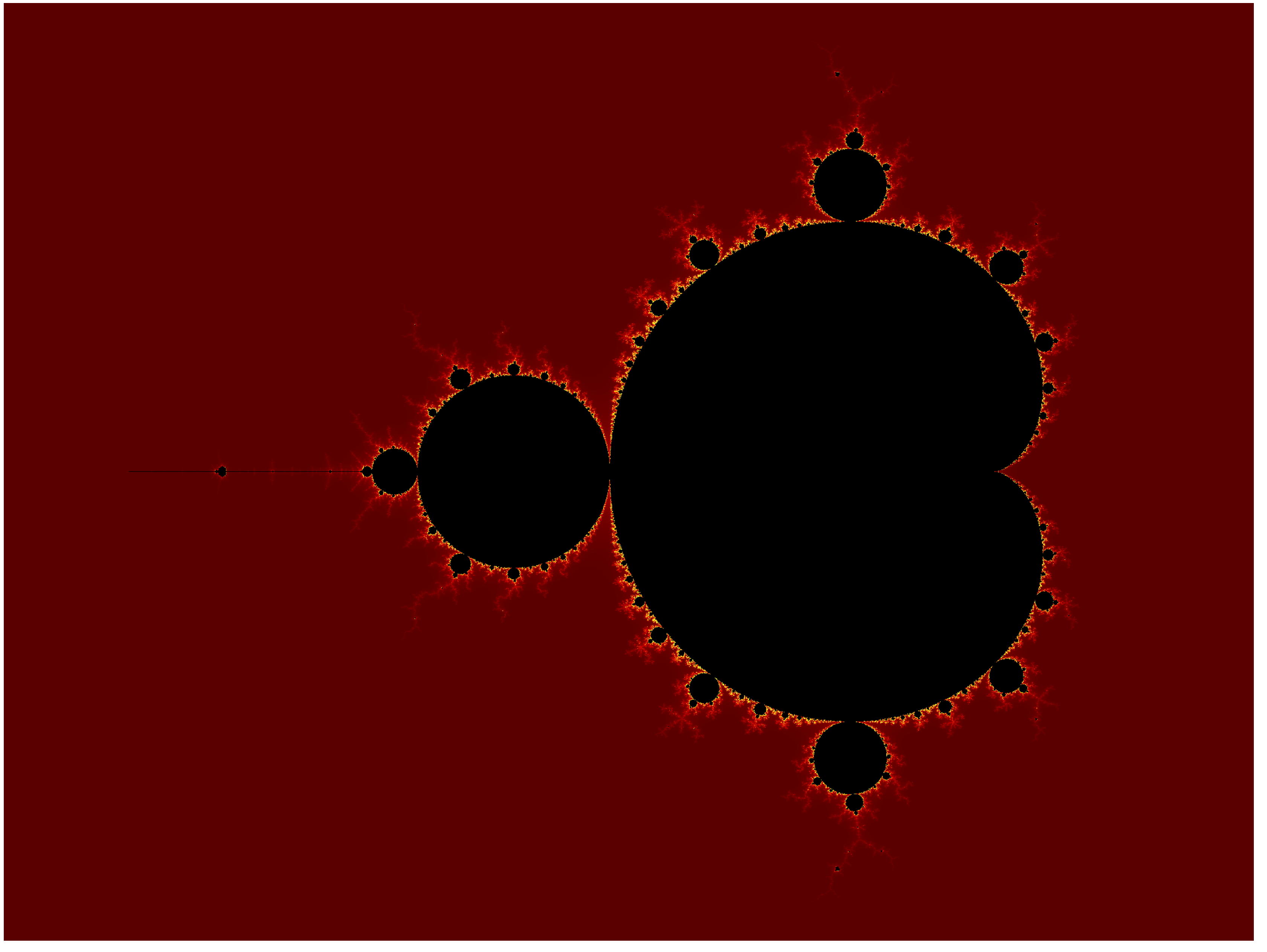


San Marco fractal is the Julia set for $f(z) = z^2 - 0.75$

St. Mark's Basilica, Venice

Mandelbrot set

DEFINITION: Mandelbrot set is the set of all c such that 0 belongs to the Fatou set of $f(z) = z^2 + c$.



Properties of Fatou and Julia sets

REMARK: Let $f: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a holomorphic map. **Then the Fatou $F(f)$ and Julia set $J(f)$ of f are f-invariant.**

LEMMA: (Iteration lemma) For each k, $J(f) = J(f^k)$, where f^k is k-th iteration of f.

Proof. Step 1: Clearly, $F(f^k) \subset F(f)$, because $\overline{\{f^k, f^{2k}, f^{3k}, ...\}}$ is compact when $\overline{\{f, f^2, f^3, ...\}}$ is compact.

Step 2: Conversely, suppose that $X = F(f^k)$; then $\overline{\{f^k, f^{2k}, f^{3k}, ...\}}$ is compact, but then $\overline{\{f, f^{k+1}, f^{2k+1}, f^{3k+1}, ...\}}$ is also compact as a continuous image of a compact (the composition is continuous in uniform topology), same for $\overline{\{f^2, f^{k+2}, f^{2k+2}, f^{3k+2}, ...\}}$, and so on. Then $\overline{\{f, f^2, f^3, ...\}}$ is obtained as a union of k compact sets. **Therefore, $F(f) \subset F(f^k)$.** ■

Properties of Fatou and Julia sets (2)

THEOREM: Julia set of polynomial map $f: \mathbb{C} \longrightarrow \mathbb{C}$ is non-empty, unless $\deg f \leqslant 1$.

Proof: Let $\Delta \subset \mathbb{C}P^1$, and $n(g)$ the number of critical points of a holomorphic function g in Δ. Then $n(g) = \frac{1}{\pi\sqrt{-1}} \int_{\partial\Delta} \frac{g'}{g} dz$, and this number is locally constant in uniform topology if g has no critical points on the boundary. Since the number of critical points of f^i is $i \deg f - 1$, it converges to infinity, hence f^i **cannot converge to a holomorphic function everywhere.** ■