Lecture 12: Fatou and Julia sets

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Uniform convergence (reminder)

DEFINITION: A sequence of maps $f_i : M \longrightarrow N$ between metric spaces **uniformly converges** (or **converges uniformly on compacts**) to $f : M \longrightarrow N$ if for any compact $K \subset M$, we have $\lim_{i\to\infty} \sup_{x\in K} d(f_i(x), f(x)) = 0$.

Claim 1: Suppose that a sequence $f_i : M \longrightarrow N$ of 1-Lipschitz maps converges to f pointwise in a countable dense subset $M' \subset M$. **Then f_i converges to f uniformly on compacts.**

COROLLARY 1: The space of Lipschitz maps **is closed in the topology of pointwise convergence.** Moreover, **pointwise convergence of Lipschitz maps implies uniform convergence on compacts.**

Arzelà-Ascoli theorem for Lipschitz maps (reminder)

DEFINITION: Let M, N be metric spaces. A subset $B \subset N$ is **bounded** if it is contained in a ball of finite radius. A family $\{f_\alpha\}$ of maps $f_\alpha : M \longrightarrow N$ is called **uniformly bounded on compacts** if for any compact subset $K \subset M$, there is a bounded subset $C_K \subset N$ such that $f_\alpha(K) \subset C_K$ for any element f_α of the family.

THEOREM: (Arzelà-Ascoli for Lipschitz maps)
Let M be a metric space and $\mathcal{F} := \{f_\alpha\}$ a set of 1-Lipschitz maps $f_\alpha : M \longrightarrow \mathbb{C}$, uniformly bounded on compacts. Assume that M has countable base of open sets and can be obtained as a countable union of compact subsets. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to a function $f : M \longrightarrow \mathbb{C}$ uniformly on compacts.**

REMARK: The limit f is also 1-Lipschitz. Indeed, a uniform convergent family is pointwise convergent, and a pointwise limit of 1-Lipschitz maps is also 1-Lipschitz (Corollary 1).

Normal families of holomorphic functions (reminder)

DEFINITION: Let M be a complex manifold. A family $\mathcal{F} := \{f_\alpha\}$ of holomorphic functions $f_\alpha : M \longrightarrow \mathbb{C}$ is called **normal family** if $\mathcal{F}$ is uniformly bounded on compact subsets.

THEOREM: (Montel's theorem)
Let M be a complex manifold with countable base, and $\mathcal{F}$ a normal, infinite family of holomorphic functions. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : M \longrightarrow \mathbb{C}$ uniformly on compacts,** and f is holomorphic.

Proof. Step 1: As in the first step of Arzelà-Ascoli, it suffices to prove Montel's theorem on a subset of M where $\mathcal{F}$ is bounded. Therefore, **we may assume that all f_α map M into a disk Δ.**

Step 2: All f_α are 1-Lipschitz with respect to Kobayashi metric. Therefore, **Arzelà-Ascoli theorem can be applied, giving a uniform limit $f = \lim f_i$.**

Step 3: A uniform limit of holomorphic functions is holomorphic by Cauchy formula. ■

REMARK: The sequence $f = \lim f_i$ converges uniformly with all derivatives, again by Cauchy formula.

Montel theorem for holomorphic maps on manifolds

The following theorem is proven by the same argument as the usual Montel theorem.

THEOREM: Let $\mathcal{F} = \{F_\alpha\}$ be a family of holomorphic maps $F: M \longrightarrow M$, where M are complex manifolds. Assume that for any compact $K \subset M$ there exists a Kobayashi hyperbolic open subset $N_K \subset N$ such that for all $F_i \in \mathcal{F}$, we have $F_i(K) \subset N_K$. **Then any sequence $\{F_i\} \subset \mathcal{F}$ has a subsequence which converges (uniformy on compacts) to a holomorphic map $F: M \longrightarrow N$.**

■

However, when N is not Kobayashi hyperbolic, convergence might fail, even it is compact.

EXAMPLE: Consider a map $F = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{C})$ acting on $\mathbb{C}P^1$. **Then $\lim_n F^n(x:y) = 1:0$ when $x \neq 0$ and $\lim_n F^n(x:y) = 0:1$ otherwise.**

Fatou and Julia sets

DEFINITION: Let X, Y be complex varieties, and $\mathcal{F}$ a family of holomorphic maps $f_\alpha: X \longrightarrow Y$. Recall that $\mathcal{F}$ is **a normal family** if any sequence $\{f_i\}$ in $\mathcal{F}$ has a subsequence which converges (uniformly on compacts) to a holomorphic map.

DEFINITION: Let $f: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a rational map, and $\{f^i\} = \{f, f \circ f, f \circ f \circ f, ...\}$ the set of all iterations of f. **Fatou set of f** is the set of all points $x \in \mathbb{C}$ such that for some neighbourhood $U \ni x$, the restriction $\{f^i|_U\}$ is a normal family, with all f^i except finitely many taking values in a disk $B \subset \mathbb{C}P^1$, and **Julia set** is the complement to Fatou set.

EXAMPLE: For the map $f(x) = x^2$, Julia set is the unit circle, and the Fatou set is its complement **(prove it)**.

Pierre Fatou and Gaston Julia

Pierre Fatou,
1878-1929

Gaston Julia
1893-1978

Pierre Fatou and Gaston Julia: the birth of holomorphic dynamics

...In 1915, the Académie des Sciences in Paris gave the topic for its 1918 Grand Prix. The prize would be awarded for a study of iteration from a global point of view. The author of [2] suggests that mathematicians such as Paul Appell, Émile Picard, and Gabriel Koenigs had put forward the idea to the Académie des Sciences because they were hoping for developments of Paul Montel's concept of normal families.

Fatou wrote a long memoir which did indeed use Montel's idea of normal families to develop the fundamental theory of iteration in 1917. Although we do not know for certain that he was intending to enter for the Grand Prix, it seems almost certain that he undertook the work with that in mind.

Given that the topic had been proposed for the prize, it is not surprising that another mathematician would also work on the topic, and indeed Gaston Julia also produced a long memoir developing the theory in a similar way to Fatou. The two, however, chose different ways to go forward. During the later half of 1917 Julia deposited his results in sealed envelopes with the Académie des Sciences. Fatou, on the other hand, published an announcement of his results in the note Sur les substitutions rationnelles in the December 1917 part of Comptes Rendus. It later became evident that they had discovered very similar results. Julia wrote a letter to Comptes Rendus concerning priority which was published on 31 December 1917. Julia had asked the Académie des Sciences to inspect his sealed envelopes and Georges Humbert had been asked to carry out the task. In the same 31 December 1917 part of Comptes Rendus Georges Humbert has a letter reporting on Julia's papers. Almost certainly as a result of these letters Fatou did not enter for the Grand Prix and it was awarded to Julia. Fatou did not lose out completely, however, and even though he had not entered for the prize, the Académie des Sciences gave him an award for his outstanding 280-page paper on the topic, "Sur les équations fonctionnelles", published in 1920...

Gaston Julia (1893-1978)

Attractor points

DEFINITION: Attractor point z is a fixed point of f such that $|df|(z) < 1$; the **attractor basin** for z is the set of all $x \in \mathbb{C}$ such that $\lim_i f^i(x) = z$.

CLAIM: For any fixed point z, **its attractor basin belongs to the Fatou set.**

Proof: Indeed, since $\lim_i f^i(x) = z$ for any point in attractor basin U, $\{f^i\}$ is a normal family on U **(pointwise convergence is equivalent to uniform convergence for bounded holomorphic functions by Arzela-Ascoli theorem).** ■

Clearly, z **is an attracting fixed point of** $f(t)$ **if** $f(z) = z$ **and** $|f'(z)| < 1$**.**

Newton iteration method

DEFINITION: Newton iteration for solving the polynomial equation $g(z) = 0$: a solution is obtained as a limit $\lim_i f^i(z)$, where $f(z) = z - \frac{g(z)}{g'(z)}$.

CLAIM: The solutions of $g(z) = 0$ are attracting fixed points of f.

Proof. Step 1: Let z be a solution of $g(z) = 0$, and m the multiplicity of zero of g in z. Since $g'(z)$ haz zero of multiplicity $m-1$, one has $\frac{g(z)}{g'(z)} = 0$, the function $\frac{g(z)}{g'(z)}$ is holomorphic in a neighbourhood of z by Riemann removable singularity theorem, and $f(z) = z$.

Step 2: To simplify the formulas, assume that $z = 0$. Since $g(0) = 0$ and g has a zero of multiplicity m, the Taylor decomposition for g in 0 takes form

$$g(z) = a_m z^m + a_{m+1} z^{m+1} + ...,$$

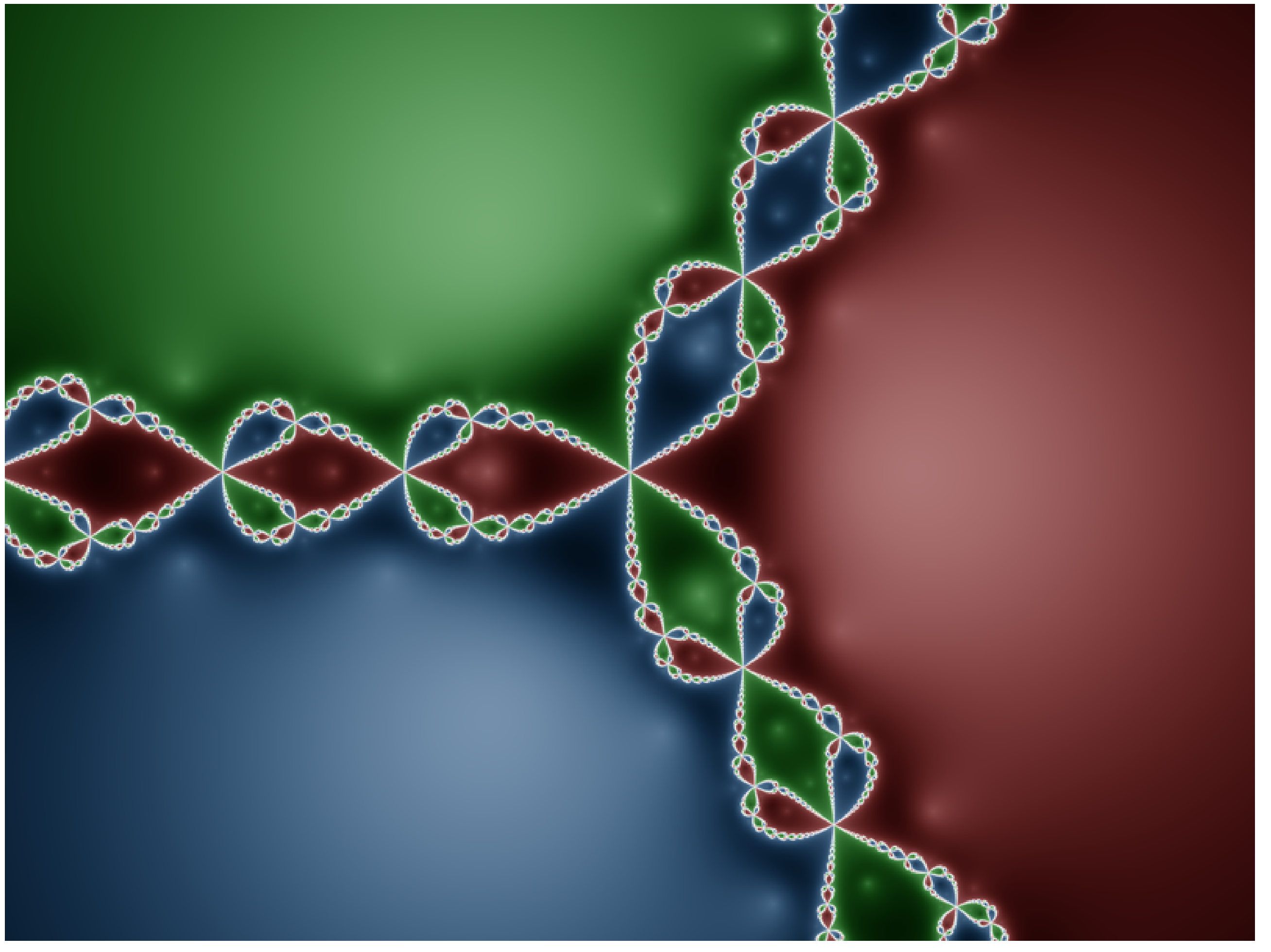
where $a_m \neq 0$. Then

$$g'(z) = m a_m z^{m-1} + (m+1) a_{m+1} z^m + ...$$

Let $u(z) = a_m + a_{m+1} z + a_{m+2} z^2 + ...$ and $v(z) = m a_m + (m+1) a_{m+1} z + (m+2) a_{m+2} z^2 + ...$ Since $\sum |a_i|$ converges, these series converge in an appropriate neighbourhood of zero. Clearly, $g(z) = z^m u(z)$ and $g'(z) = z^{m-1} u(z)$, which gives $\frac{g(z)}{g'(z)} = z \frac{u(z)}{v(z)}$, and $\frac{d}{dz} \frac{g(z)}{g'(z)}|_{z=0} = \frac{u(0)}{v(0)} = \frac{1}{m}$. This gives $f'(0) = 1 - \frac{1}{m}$. ■

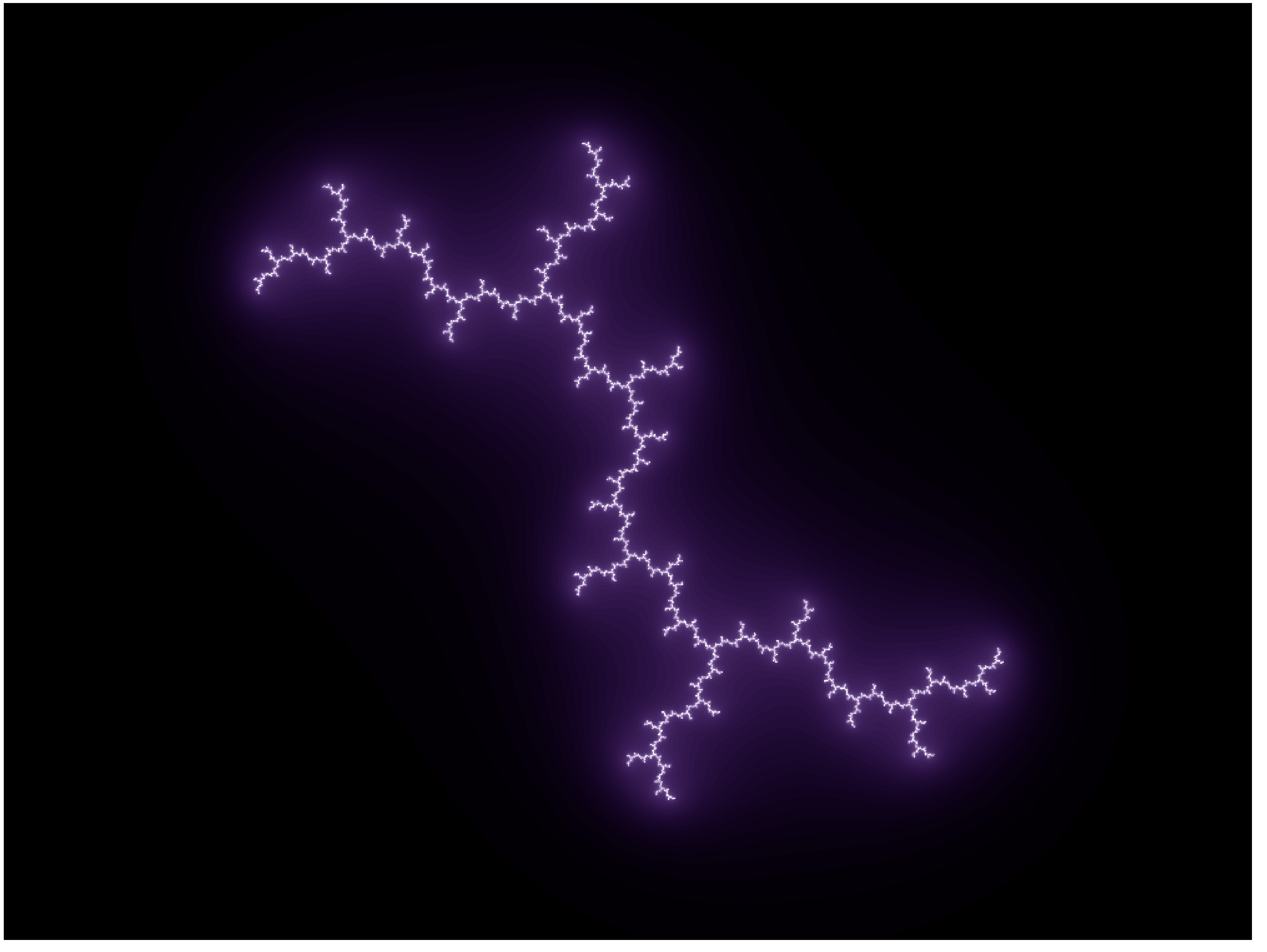
Fatou and Julia sets for $f(z) = \frac{1+2z^3}{3z^2}$

We apply the Newton iteration method to $g(z) = z^3 - 1$.



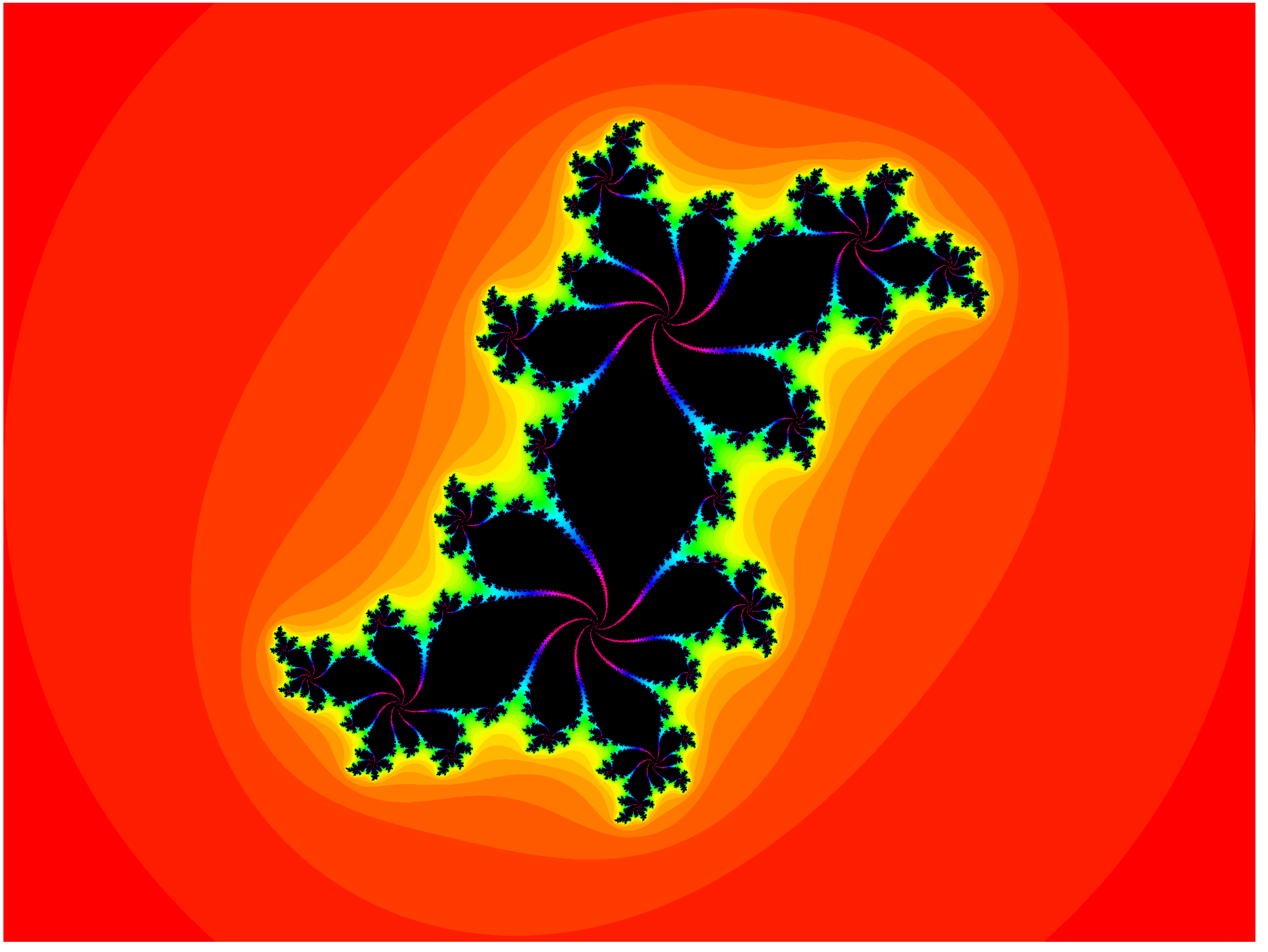
Julia set (in white) for the map $f(z) = \frac{1+2z^3}{3z^2} = z - \frac{g(z)}{g'(z)}$. Attractor basins for three roots of $g(z) = z^3 - 1$ are colored in red, green, blue.

Julia set for $f(z) = z^2 - \sqrt{-1}$

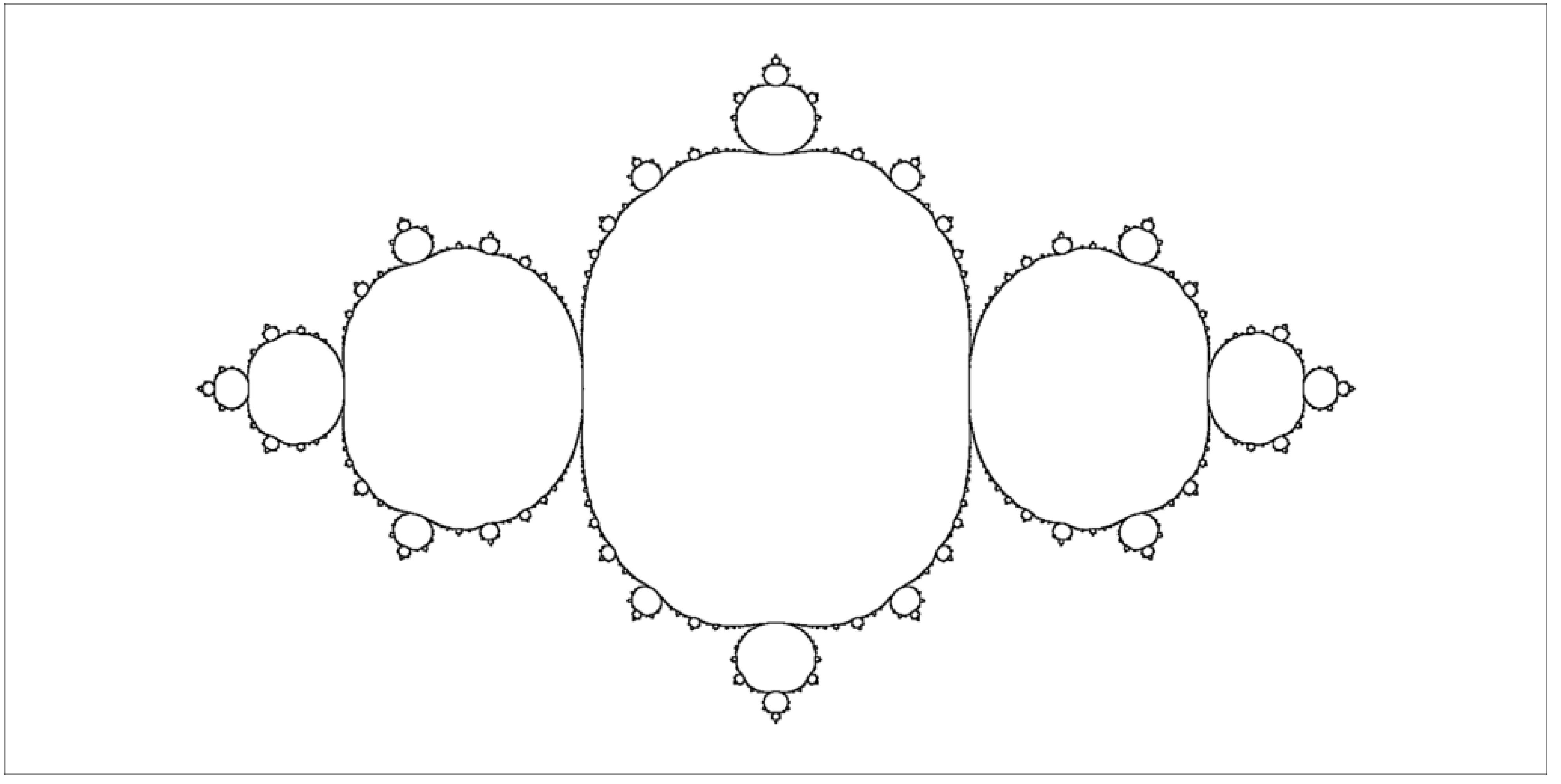


Julia set for $f(z) = z^2 - \sqrt{-1}$ is called **dendrite**.

Julia set for $f(z) = z^2 + 0.12 + 0.6\sqrt{-1}$



San Marco fractal

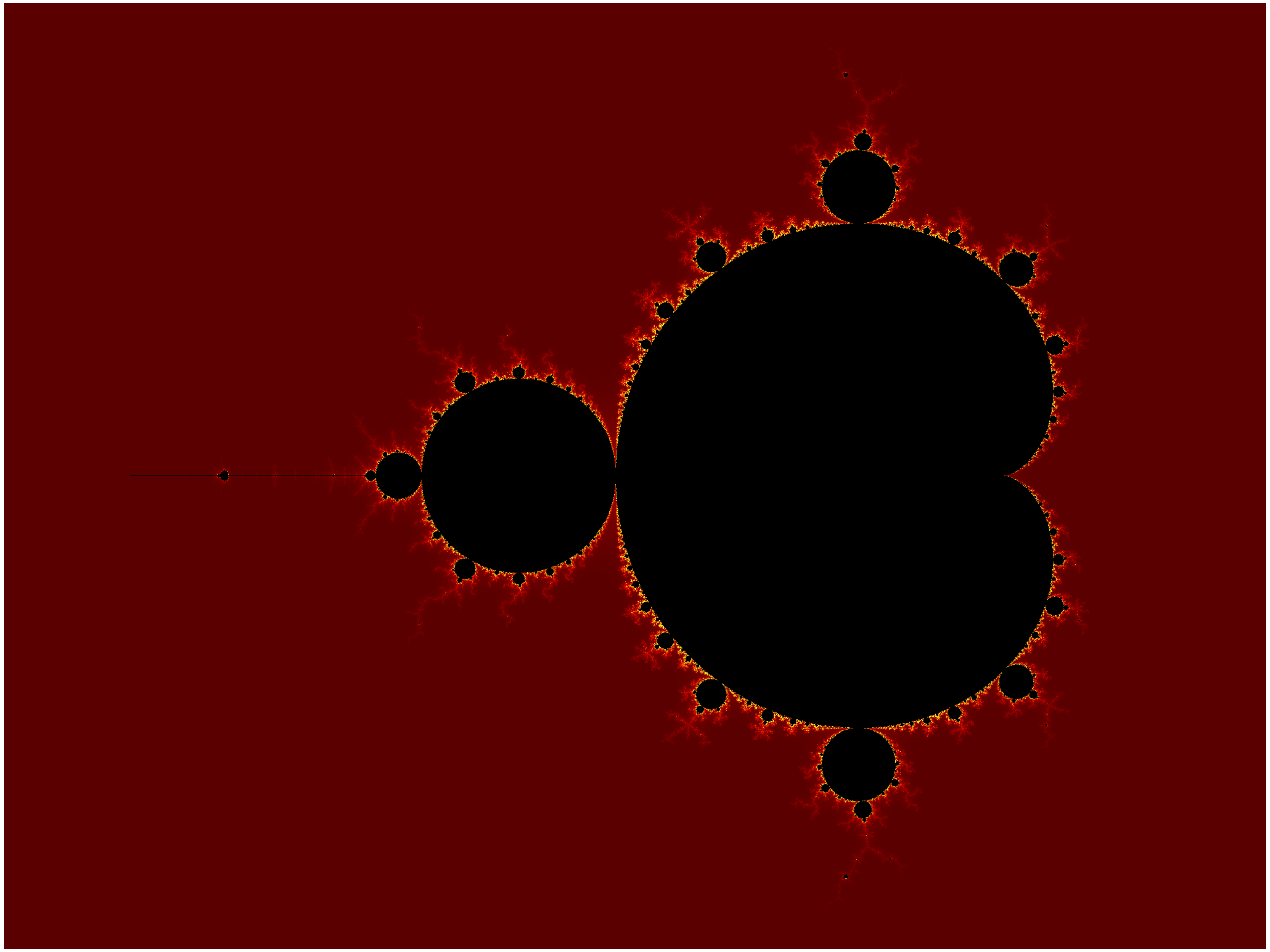


San Marco fractal is the Julia set for $f(z) = z^2 - 0.75$

St. Mark's Basilica, Venice

Mandelbrot set

DEFINITION: Mandelbrot set is the set of all c such that 0 belongs to the Fatou set of $f(z) = z^2 + c$.



Properties of Fatou and Julia sets

REMARK: Let $f: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a holomorphic map. **Then the Fatou $F(f)$ and Julia set $J(f)$ of f are f-invariant.**

LEMMA: (Iteration lemma) For each k, $J(f) = J(f^k)$, where f^k is k-th iteration of f.

Proof. Step 1: Clearly, $F(f^k) \subset F(f)$, because $\overline{\{f^k, f^{2k}, f^{3k}, ...\}}$ is compact when $\overline{\{f, f^2, f^3, ...\}}$ is compact.

Step 2: Conversely, suppose that $X = F(f^k)$; then $\overline{\{f^k, f^{2k}, f^{3k}, ...\}}$ is compact, but then $\overline{\{f, f^{k+1}, f^{2k+1}, f^{3k+1}, ...\}}$ is also compact as a continuous image of a compact (the composition is continuous in the topology of uniform convergence on compacts), same for $\overline{\{f^2, f^{k+2}, f^{2k+2}, f^{3k+2}, ...\}}$, and so on. Then $\overline{\{f, f^2, f^3, ...\}}$ is obtained as a union of k compact sets. **Therefore, $F(f) \subset F(f^k)$.** ■

Properties of Fatou and Julia sets (2)

THEOREM: Julia set of polynomial map $f: \mathbb{C} \longrightarrow \mathbb{C}$ is non-empty, unless $\deg f \leqslant 1$.

Proof: Let $\Delta \subset \mathbb{C}P^1$ be a disk, and $n(g)$ the number of critical points of a holomorphic function g in Δ. Then $n(g) = \frac{1}{\pi\sqrt{-1}} \int_{\partial\Delta} \frac{g''}{g'} dz$, and this number is locally constant in uniform topology if g has no critical points on the boundary. Since the number of critical points of f^i is $i(\deg f - 1)$, it converges to infinity, and there is an accumulation point. Take a disk Δ containing this point, and such that all f^i have no critical points on its boundary. Then the integral $\frac{1}{\pi\sqrt{-1}} \int_{\partial\Delta} \frac{(f^i)''}{(f^i)'} dz$ converges to infinity, hence **f^i cannot converge to a holomorphic function everywhere.** ∎