Lecture 13: Ramified coverings

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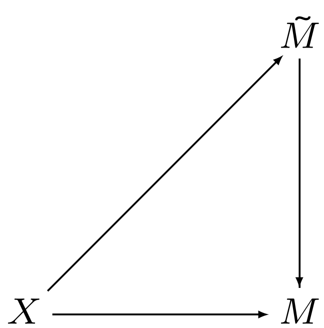
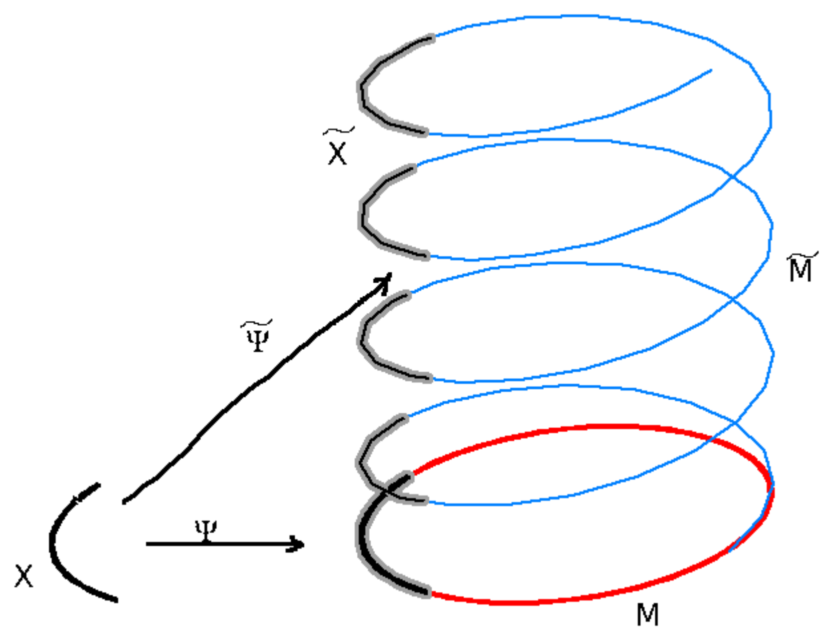
Homotopy lifting principle (reminder)

DEFINITION: A topological space X is **locally path connected** if for each $x \in X$ and each neighbourhood $U \ni x$, there exists a smaller neighbourhood $W \ni x$ which is path connected.

THEOREM: (homotopy lifting principle)
Let X be a simply connected, locally path connected topological space, and $\tilde{M} \longrightarrow M$ a covering map. Then for each continuous map $X \longrightarrow M$, there exists a lifting $X \longrightarrow \tilde{M}$ making the following diagram commutative.





Coverings and subgroups of $\pi_1(M)$

THEOREM: For each subgroup $\Gamma \subset \pi_1(M)$ **there exists a unique, up to isomorphism, connected covering $M_\Gamma \longrightarrow M$ such that $\pi_1(M_\Gamma) = \Gamma$.**

THEOREM: If, in addition, $\Gamma \subset \pi_1(M)$ is a normal subgroup, **the group $G = \pi_1(M)/\Gamma$ acts on M_Γ by automorphisms** commuting with projection to M ("automorphisms of the covering"), freely and transitively on the fibers of the projection $M_\Gamma \longrightarrow M$, and give $M = M_\Gamma/G$.

COROLLARY: The fundamental group $\pi_1(M)$ **acts on the universal covering $\tilde{M}$ by homeomorphisms which commute with the projection to M and give $M = \tilde{M}/\pi_1(M)$.**

THEOREM: Let M be connected, locally path connected, locally simply connected topological space. Fix $p \in M$. Then **the category of the coverings of M is naturally equivalent with the category of sets with the action of $\pi_1(M)$,** and the equivalence takes a covering $\tilde{M} \longrightarrow M$ to the set $\pi^{-1}(p)$.

COROLLARY: Let M be a space with commutative $\pi_1(M)$, and $\tilde{M}$ its universal cover. **Then for any connected covering $M_1 \longrightarrow M$, the covering M_1 is obtained as $M_1 = \tilde{M}/\Gamma$, where $\Gamma \subset \pi_1(M)$ is a subgroup.**

Deformation retracts

DEFINITION: Retraction of a topological space X to $Y \subset X$ is a continuous map $X \longrightarrow Y$ which is identity on $Y \subset X$. **Deformation retraction** of a topological space X to $Y \subset X$ is a continuous map $\varphi_t : X \times [0,1] \longrightarrow X$ such that $\varphi_1 = \mathrm{Id}_X$ and φ_0 is retration of X to Y.

EXERCISE: Prove that $\pi_1(X) = \pi_1(Y)$ when Y is a deformation retract of X.

DEFINITION: A topological space X is **contractible** if a point $p \in X$ is its deformation retract.

EXERCISE: Let $p \in X$ be a deformation retract of X, **prove that any other point $q \in X$ is also a deformation retract.**

EXERCISE: Prove that **a contractible space X satisfies $\pi_1(X) = 0$.**

EXERCISE: Let $Y \subset X$ be a deformation retract of X. **Prove that any map $Z \longrightarrow X$ is homotopy equivalent to $Z \longrightarrow Y \subset X$.**

Points of ramification

DEFINITION: Let $\varphi: X \longrightarrow Y$ be a holomorphic map of complex 1-dimensional manifolds, not constant on each connected component of X. Any point $x \in X$ where $d\varphi = 0$ is called **a ramification point** of φ. **Ramification index** of the point x is the number of preimages of $y' \in Y$ in U_x, where U_x is a small neighbourhood of $x \in X$, $U_y :== \varphi(U_x)$, and $y' \in U_y$ is a general point.

THEOREM 1: Let Y, Y be compact Riemannian surfaces, $\varphi: X \longrightarrow Y$ a holomorphic map, and $x \in X$ a ramification point. Then there is a neighbourhood of $x \in X$ biholomorphic to a disk Δ, **such that the map $\varphi|_\Delta$ is equivalent to $\varphi(x) = x^n$,** where n is the ramification index.

Proof. Step 1: Let $W \subset Y$ be a sufficiently small simply connected neighbourhood of $y \in Y$, and $U \ni x$ a connected component of its preimage in X. Choosing W sufficiently small, we may assume that U lies in a coordinate chart. The zeros of $d\varphi$ are isolated. Shrinking W if nessesarily, we may assume that $d\varphi$ is nowhere zero on $U \backslash x$, and $U \backslash x \stackrel{\varphi}{\longrightarrow} W \backslash y$ is a covering. We identify W with a disk Δ. By homotopy lifting principle, the homothety map $\lambda \longrightarrow r\lambda$ of W, $r \in [0, 1]$ can be lifted to U uniquely. **This means that x is a homotopy retract of U, and $\pi_1(U) = 0$.** Riemann mapping theorem implies that U **is isomorphic to a disk.**

Points of ramification (2)

THEOREM 1: Let Y, Y be compact Riemannian surfaces, $\varphi: X \longrightarrow Y$ a holomorphic map, and $x \in X$ a ramification point. Then there is a neighbourhood of $x \in X$ biholomorphic to a disk Δ, **such that the map $\varphi|_\Delta$ is equivalent to $\varphi(x) = x^n$,** where n is the ramification index.

Proof. Step 1: Let $W \subset Y$ be a sufficiently small simply connected neighbourhood of $y \in Y$, and $U \ni x$ a connected component of its preimage in X. [...] Choose U, W in such a way that that x is a homotopy retract of U, and $\pi_1(U) = 0$. **Riemann mapping theorem implies that U is isomorphic to a disk.**

Step 2: Passing to the universal covering $\widetilde{U \backslash x} = \widetilde{W \backslash y}$, we obtain an holomorphic action of $\mathbb{Z} = \pi_1(\widetilde{U \backslash x})$ on $\widetilde{W \backslash y}$ such that $W \backslash y = \widetilde{W \backslash y}/\mathbb{Z}$ and $U \backslash y = \widetilde{W \backslash y}/n\mathbb{Z}$. Therefore, $\mathbb{Z}/n\mathbb{Z}$ acts on $U \backslash x$, freely and transitively on the fibers of the projection $U \backslash x \xrightarrow{\varphi} W \backslash y$. This action is extended to 0 by homotopy lifting principle. **Then $W = U/(\mathbb{Z}/n)$.** However, **any action of the cyclic group $\mathbb{Z}/n$ on Δ is conjugate to the rotations by $\{\varepsilon_n^i\}$,** where ε_n is a primitive root of unity of degree n. The corresponding quotient map is equivalent to $\varphi(x) = x^n$. ■

Hyperelliptic curves and hyperelliptic equations

REMARK: Let $M \stackrel{\Psi}{\longrightarrow} N$ be a C^∞-map of compact smooth oriented manifolds. Recall that **degree** of Ψ is number of preimages of a regular value n, counted with orientation. Recall that **the number of preimages is independent from the choice of a regular value $n \in N$**, and **the degree is a homotopy invariant.**

DEFINITION: Hyperelliptic curve S is a compact Riemann surface admitting a holomorphic map $S \longrightarrow \mathbb{C}P^1$ of degree 2 and with $2n$ ramification points of index 2.

DEFINITION: Hyperelliptic equation is an equation $P(t,y) = y^2 + F(t) = 0$, where $F \in \mathbb{C}[t]$ is a polynomial with no multiple roots.

REMARK: Clearly, the natural projection $(t,y) \longrightarrow t$ maps the set S_0 of solutions of $P(t,y) = 0$ to $\mathbb{C}$ with $2n$ ramification points of index 2. Also, S_0 is smooth **(check this).** The complex manifold S_0 **is equipped with an involution $\tau(t,y) = (t,-y)$ exchanging the roots, and $S_0/\tau = \mathbb{C}$.**

Hyperelliptic curves and desingularization

DEFINITION: Let $P(t,y) = y^2 + F(t) = 0$ be a hyperelliptic equation. **Homogeneous hyperelliptic equation** is $P(x,y,z) = y^2 z^{n-2} + z^n F(x/z) = 0$, where $n = \deg F$.

REMARK: The set of solutions of $P(x,y,z) = 0$ is singular, but an algebraic variety of dimension 1 has a natural desingularization, called **normalization**. **The involution τ is extended to the desingularization S,** giving $S/\tau = \mathbb{C}P^1$ because $\mathbb{C}P^1$ is the only smooth holomorpic compactification of $\mathbb{C}$ as we have seen already.

Hyperbolic polyhedral manifolds

DEFINITION: A **polyhedral manifold** of dimension 2 is a piecewise smooth manifold obtained by gluing polygons along edges.

DEFINITION: Let $\{P_i\}$ be a set of polygons on the same hyperbolic plane, and M be a polyhedral manifold obtained by gluing these polygons. Assume that all edges which are glued have the same length, and we glue the edges of the same length. Then M is called **a hyperbolic polyhedral manifold**. We consider M as a metric space, with the path metric induced from P_i.

CLAIM: Let M be a hyperbolic polyhedral manifold. Then for each point $x \in M$ which is not a vertex, x **has a neighbourhood which is isometric to an open set of a hyperbolic plane.**

Proof: For interior points of M this is clear. When x belongs to an edge, it is obtained by gluing two polygons along isometric edges, hence the neighbourhood is locally isometric to the union of the same polygons in $\mathbb{H}^2$ aligned along the edge. ■

Hyperbolic polyhedral manifolds: interior angles of vertices

DEFINITION: Let $v \in M$ be a vertex in a hyperbolic polyhedral manifold. **Interior angle** of v in M is sum of the adjacent angles of all polygons adjacent to v.

EXAMPLE 1: Let $M \longrightarrow \Delta$ be a ramified n-tuple cover of the Policate disk, given by solutions of $y^n = x$. We can lift split Δ to polygons and lift the hyperbolic metric to M, obtaining M as a union of n times as many polygons glued along the same edges. **Then the interior angle of the ramification point is $2\pi n$.**

EXAMPLE 2: Let $\Delta \longrightarrow M$ be a ramified n-tuple cover, obtained as a quotient $M = \Delta/G$, where $G = \mathbb{Z}/n\mathbb{Z}$. Split Δ onto fundamental domains of G, shaped like angles adjacent to 0. Then the quotient Δ/G gives an angle with its opposite sides glued. **It is a hyperbolic polyhedral manifold with interior angle $\frac{2\pi}{n}$ at its ramification point.**

EXAMPLE 3: Let D be a diameter bisecting a disk Δ, and passing through the origin 0 and $P \subset \Delta$ one of the halves. The (unique) edge of P is split onto two half-geodesics E_+ and E_- by the origin. **Gluing E_+ and E_-, we obtain a hyperbolic polyhedral manifold with a single vertex and the interior angle π.**

Sphere with n points of angle π

EXAMPLE: Let P be a bounded convex polygon in $\mathbb{H}^2$, α the sum of its angles, and a_i, $i = 1, ..., m$ median points on its edges E_i. Each a_i splits E_i in two equal intervals. We glue them as in Example 3, and glue all vertices of P together. **This gives a sphere M with hyperbolic polyhedral metric,** one vertex ν with angle α (obtained by gluing all vertices of P together) and m vertices with angle π corresponding to $a_i \in E_i$.

REMARK: Assume that $\alpha = 2\pi$, that is, M is isometric to a hyperbolic 2-ball of radius r around ν. We equip M with a complex structure compatible with the hyperbolic metric outside of its singularities. A neighbourhood of each singularity is isometrically identified with a neighbourhood of 0 in Δ/G, where $G = \mathbb{Z}/2\mathbb{Z}$. We put a complex structure on Δ/G as in Example 2. **This puts a structure of a complex manifold on M.**

THEOREM: (Alexandre Ananin)
Let M be the hyperbolic polyhedral manifold obtained from the polyhedron P as above. Assume that $m = n$ is even, and $\alpha = 2\pi$. **Then M admits a double cover M_1, ramified at all a_i, which is locally isometric to $\mathbb{H}^2$.**

Proof in the next lecture.

REMARK: Clearly, **M_1 is hyperelliptic.**