Lecture 14: Tilings and polyhedral hyperbolic manifolds

Misha Verbitsky

IMPA, sala 236, 17:00

May 6, 2024

Points of ramification (different proof)

DEFINITION: Let $\varphi : X \longrightarrow Y$ be a holomorphic map of complex manifolds of dimension 1, not constant on each connected component of X. Any point $x \in X$ where $d\varphi = 0$ is called **a ramification point** of φ. **Ramification index** of the point x is the number of preimages of $y' \in Y$, for y' in a sufficiently small neighbourhood of $y = \varphi(x)$.

THEOREM 0: Let X, Y be compact Riemann surfaces, $\varphi : X \longrightarrow Y$ a holomorphic map, and $x \in X$ a ramification point. Then there is a neighbourhood of $x \in X$ biholomorphic to a disk Δ, **such that the map $\varphi|_\Delta$ is equivalent to $\varphi(x) = x^n$,** where n is the ramification index.

Proof: Take neighbourhoods $U \ni x$, $V \ni \varphi(x)$ which are biholomorphic to a disk, with $\varphi(U) \subset V$. Write the Taylor decomposition for φ in 0: $\varphi(x) = a_n x^n + a_{n+1} x^{n+1} + ... = x^n u(x)$, where $a_n \neq 0$. where $u(x) = a_n + a_{n+1} x + a_{n+1} x^2 + ...$ Since $u(x)$ is invertible, one can choose a branch $v(x) := \sqrt[n]{u(x)}$ in a neighbourhood of 0. Then $\varphi(x) = z^n$, where $z = xv(x)$. ■

Hyperelliptic curves and hyperelliptic equations (reminder)

REMARK: Let $M \xrightarrow{\Psi} N$ be a C^∞-map of compact smooth oriented manifolds. Recall that **degree** of Ψ is number of preimages of a regular value n, counted with orientation. Recall that **the number of preimages is independent from the choice of a regular value $n \in N$,** and **the degree is a homotopy invariant.**

DEFINITION: **Hyperelliptic curve** S is a compact Riemann surface admitting a holomorphic map $S \longrightarrow \mathbb{C}P^1$ of degree 2 and with $2n$ ramification points of degree 2. **Hyperelliptic equation** is an equation $P(t,y) = y^2 + F(t) = 0$, where $F \in \mathbb{C}[t]$ is a polynomial with no multiple roots.

DEFINITION: Let $P(t,y) = y^2 + F(t) = 0$ be a hyperelliptic equation. **Homogeneous hyperelliptic equation** is $P(x,y,z) = y^2 z^{n-2} + z^n F(x/z) = 0$, where $n = \deg F$.

REMARK: The set of solutions of $P(x,y,z) = 0$ is usually singular, but an algebraic variety of dimension 1 has a natural desingularization, called **normalization**. Define the involution $\tau(x,y,z) = (x,-y,z)$. Clearly, $\tau(S) = S$. **The involution τ is extended to the desingularization S,** giving $S/\tau = \mathbb{C}P^1$ because $\mathbb{C}P^1$ is the only smooth holomorpic compactification of $\mathbb{C}$ as we have seen already.

Hyperbolic polyhedral manifolds (reminder)

DEFINITION: A **polyhedral manifold** of dimension 2 is a piecewise smooth manifold obtained by gluing polygons along edges.

DEFINITION: Let $\{P_i\}$ be a set of polygons on the same hyperbolic plane, and M be a polyhedral manifold obtained by gluing these polygons. Assume that all edges which are glued have the same length, and we glue the edges of the same length. Then M is called **a hyperbolic polyhedral manifold**. We consider M as a metric space, with the path metric induced from P_i.

CLAIM: Let M be a hyperbolic polyhedral manifold. Then for each point $x \in M$ which is not a vertex, x **has a neighbourhood which is isometric to an open set of a hyperbolic plane.**

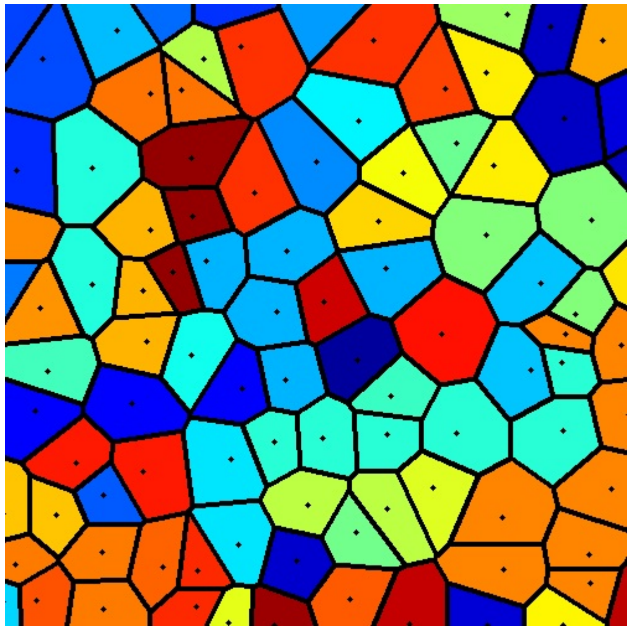
Proof: For interior points of M this is clear. When x belongs to an edge, it is obtained by gluing two polygons along isometric edges, hence the neighbourhood is locally isometric to the union of the same polygons in $\mathbb{H}^2$ aligned along the edge. ■

Voronoi partitions

DEFINITION: Let M be a metric space, and $S \subset M$ a discrete subset. **Voronoi cell** D_i associated with $x_i \in S$ is

$$D_i := \{z \in M \ \mid d(z, x_i) \leqslant d(z, x_i) \forall j \neq i\}.$$

Voronoi partition (also known as "Dirichlet tessellation") is partition of M onto its Voronoi cells.



Voronoi partition

DEFINITION: Recall that **the absolute** Abs is the infinity boundary of the Poincaré disk, identified with the projectivization of the isotropic cone in $R^{1,2}$

Fundamental domains and polygons

DEFINITION: Let Γ be a discrete group acting on a manifold M, and $U \subset M$ an open subset with piecewise smooth boundary. Assume that for any non-trivial $\gamma \in \Gamma$ one has $U \cap \gamma(U) = \emptyset$ and $\Gamma \cdot \overline{U} = M$, where $\overline{U}$ is closure of U. Then $\overline{U}$ is called **a fundamental domain** of the action of Γ.

THEOREM 1: Let Γ be a discrete group acting on a hyperbolic plane $\mathbb{H}^2$ by oriented isometries. **Then Γ has a polyhedral fundamental domain P.** If, moreover, $\mathbb{H}^2/\Gamma$ has finite volume, ∂P has at most finitely many points on the absolute Abs.

Proof. Step 1: Clearly, $\mathrm{Vol}(P) = \mathrm{Vol}(\mathbb{H}^2/\Gamma)$. This takes care of the last assertion, because polygons with infinitely many points on Abs have infinite volume.

Step 2: Let $\Gamma \cdot x$ be an orbit of Γ in $\mathbb{H}^2$. The Voronoi partition gives a polygonal fundamental domain P for the Γ-action. Clearly, $\mathbb{H}^2$ is tiled by the Voronoi cells, which are all isometric polygons. However, **the cells are not necessarily fundamental domains:** each cell P can be possibly stabilized by a subgroup $\Gamma_P \subset \Gamma$. Clearly, each cell P contains a unique point $x_1 \in \Gamma \cdot x$, hence Γ_P fixes x_1. The stabilizer of x_1 in $SO^+(1,2)$ is a circle, and Γ_P is discrete, hence it is a cyclic group, and P is a polygon admitting an action of a cyclic group. Cutting P onto angles which are freely rotated by Γ_P, we obtain a polygonal fundamental domain. ■

Group quotients and polyhedral hyperbolic manifolds

THEOREM: Let $\Gamma \subset SO^+(1,2)$ be a discrete subgroup, and $\mathbb{H}^2/\Gamma$ the quotient. **Then $\mathbb{H}^2/\Gamma$ is isometric to a polyhedral hyperbolic manifold.**

Proof. Step 1: First, **we prove that the quotient $\mathbb{H}^2/\Gamma$ is a manifold.** Indeed, outside of the set of fixed points, the action of Γ is properly discontinuous, and the quotient is smooth. For each fixed point p, it contains a neighbourhood where $\mathrm{St}_\Gamma(p)$ acts as a finite order rotation group $\mathbb{Z}/n\mathbb{Z}$ on a disk, and the quotient is also a disk.

Step 2: By Theorem 1, Γ-action **admits a polygonal fundamental domain D.** Then $\mathbb{H}^2/\Gamma$ is obtained by gluing isometric edges of D; since it is a manifold (Step 1), it is a hyperbolic polyhedral manifold. ■

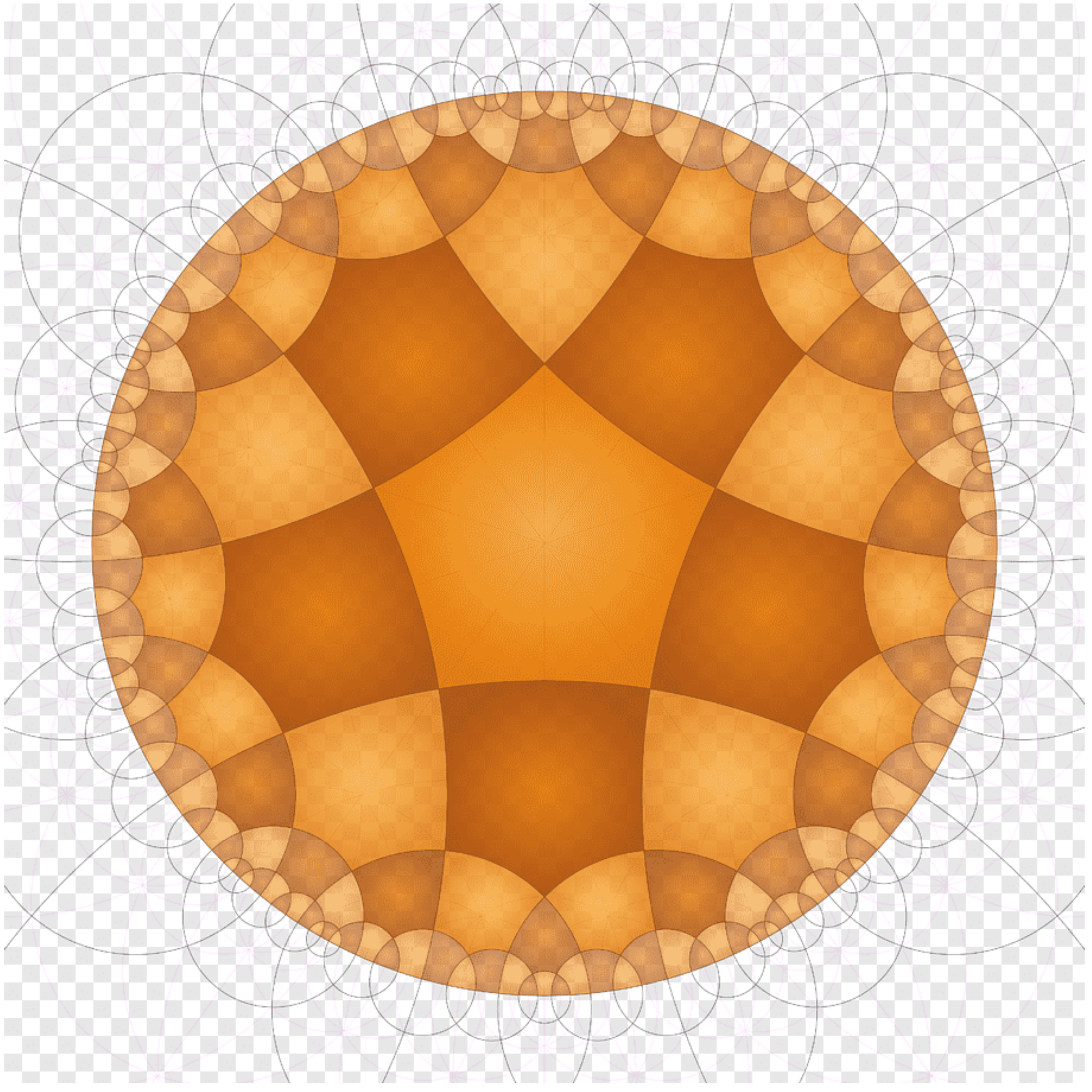
Semi-regular tilings

DEFINITION: A tiling of $\mathbb{H}^2$ is a partition of $\mathbb{H}^2$ onto polygons (or, sometimes, other figures) with finite volume. A tiling is **regular** if the group Γ of isometries preserving tilings acts transitively on vertices, edges and faces of the partition. A tiling T is **semi-regular** if Γ acts on the set of faces of T with finitely many orbits.

REMARK: Tilings is good a way to produce hyperbolic manifolds and Riemannian surfaces from a hyperbolic plane. Indeed, for any semi-regular tiling, T, **the quotient space $\mathbb{H}^2/\Gamma$ has finite volume.** Moreover, **$\mathbb{H}^2/\Gamma$ is compact if all polygons in T have no vertices in** Abs **(prove it)**.

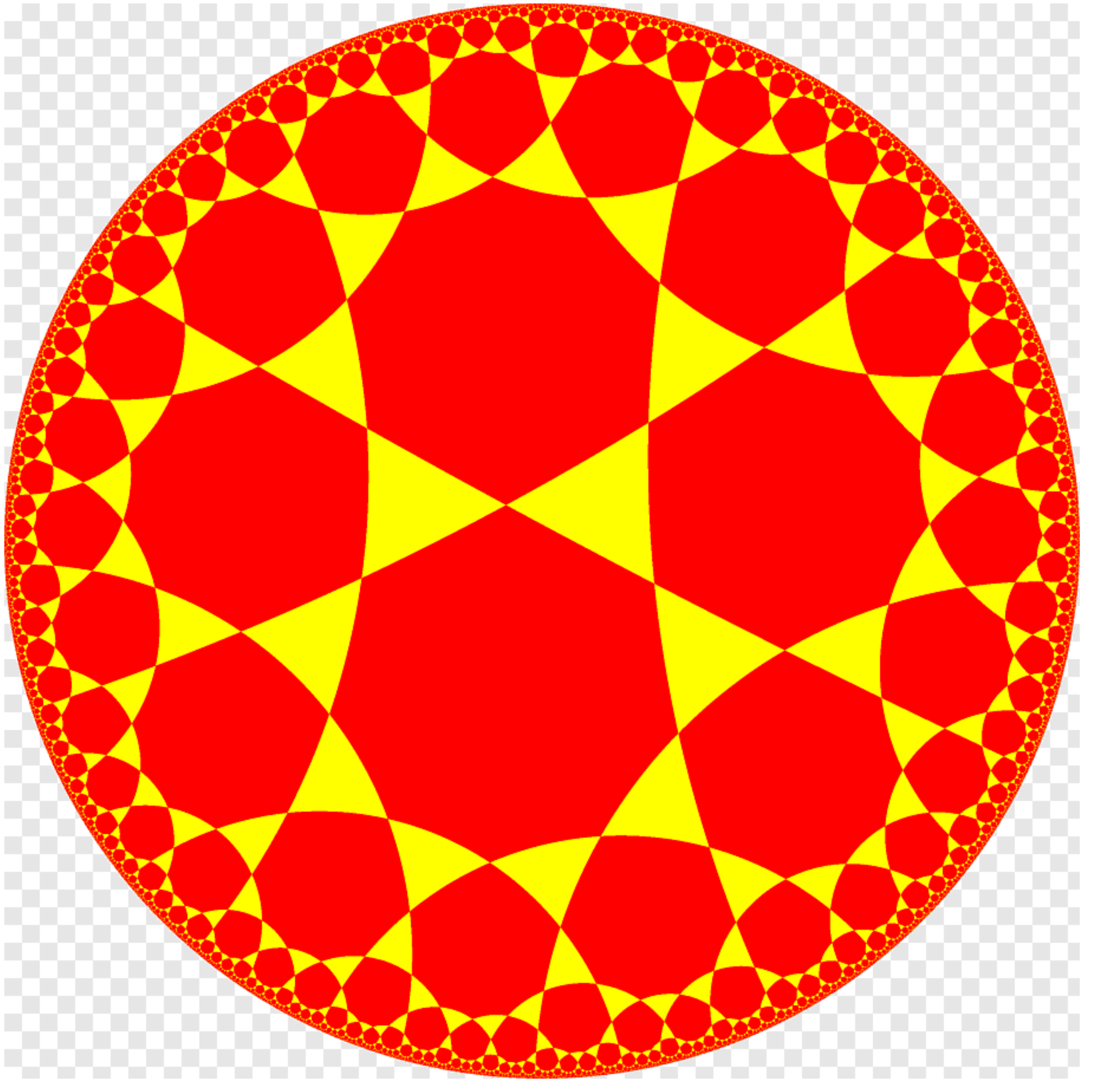
EXERCISE: Let T be a regular tiling of $\mathbb{H}^2$, and Γ the group of isometries of $\mathbb{H}^2$ preserving T. **Prove that any face of T is a fundamental domain for Γ.**

Regular tiling of $\mathbb{H}^2$ by right-angle pentagons



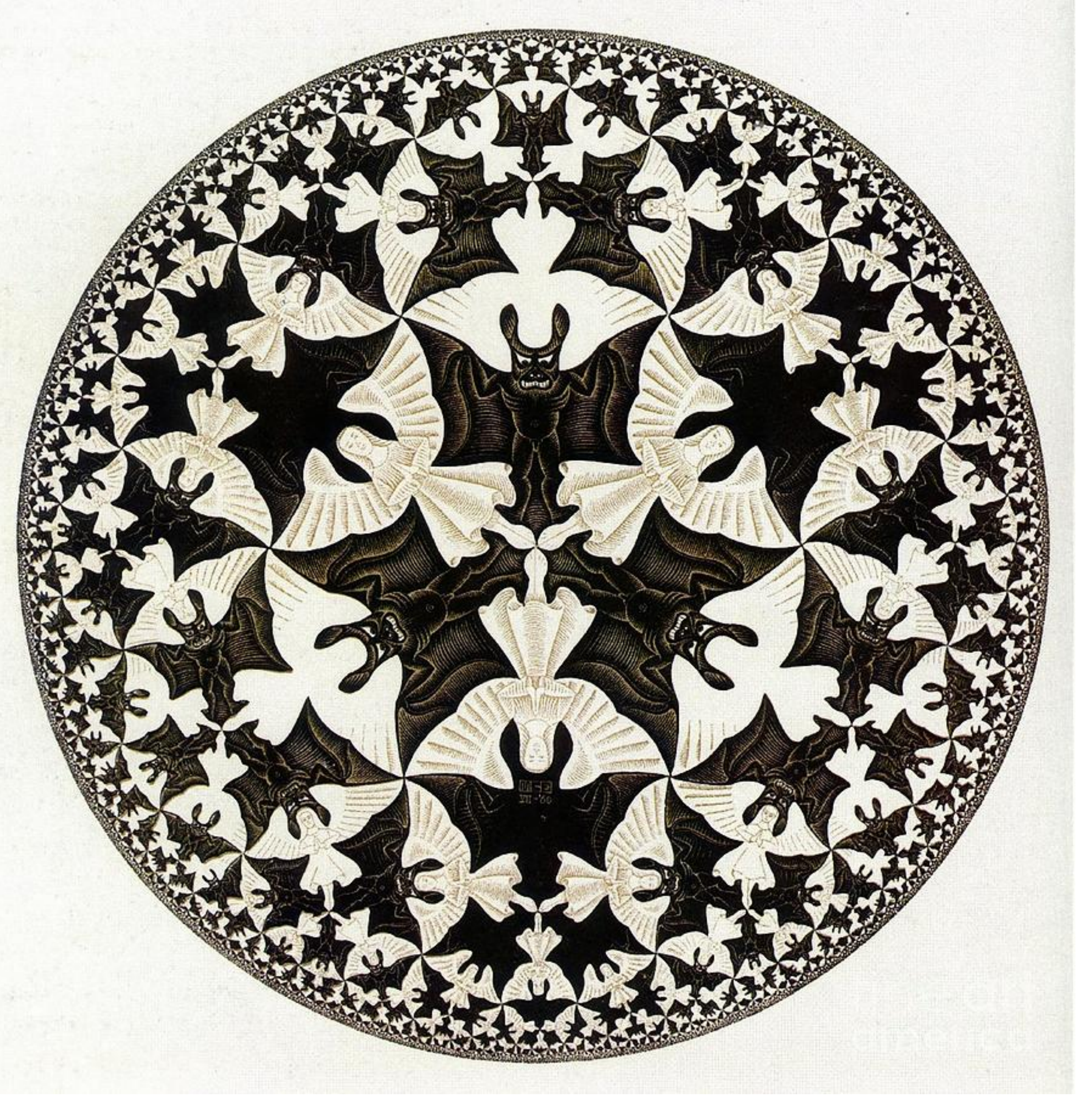
Regular tiling of $\mathbb{H}^2$ by right-angle pentagons

Semi-regular tiling of $\mathbb{H}^2$



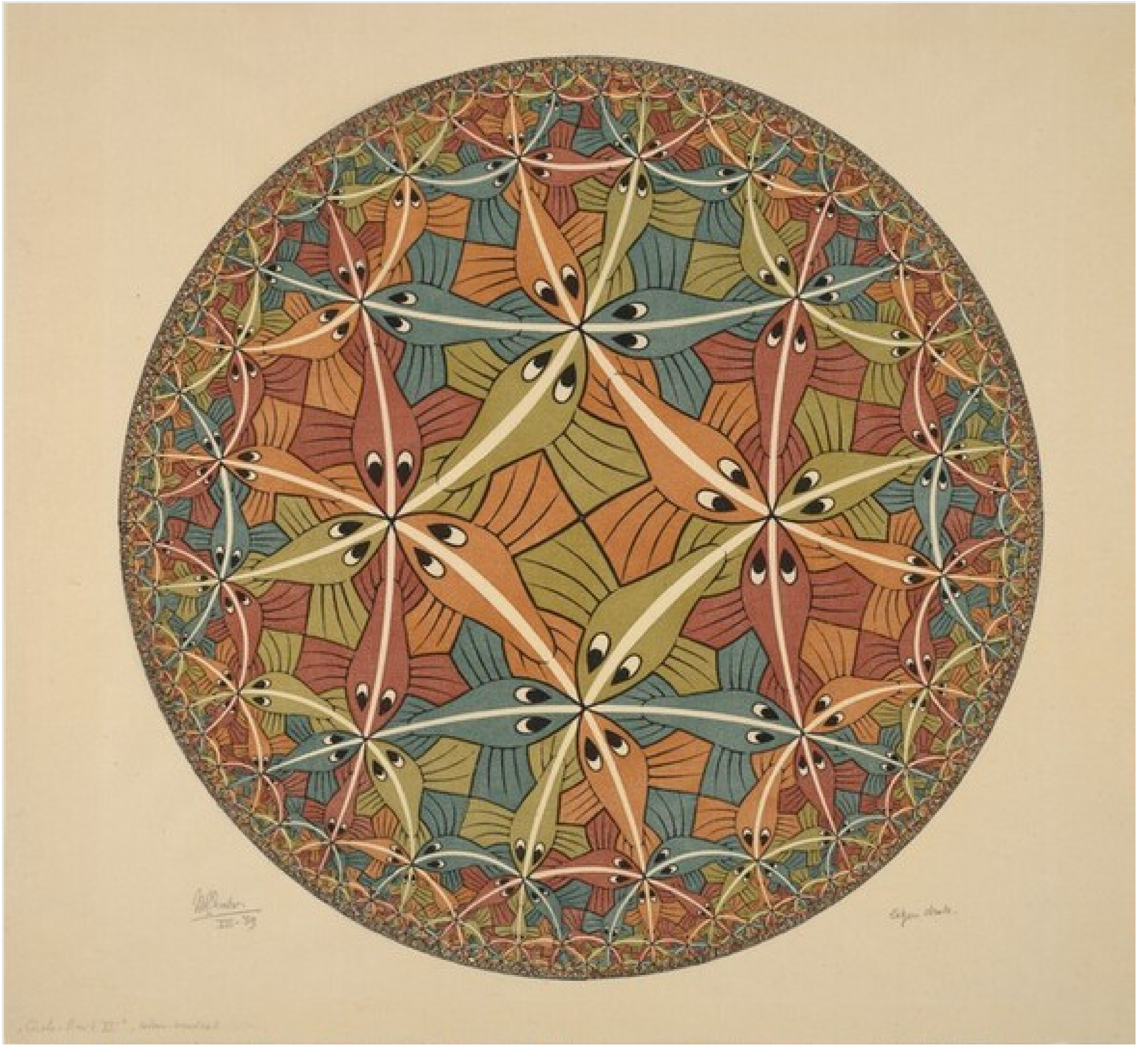
Semi-regular tiling of $\mathbb{H}^2$ by octagons and triangles

M. C. Escher, Circle Limit IV, 1960



This is an example of semi-regular tiling of a hyperbolic plane by angels and demons. In the hypebolic plane, **the angels and the demons are isometric to each other.**

M. C. Escher, Circle Limit III, 1959

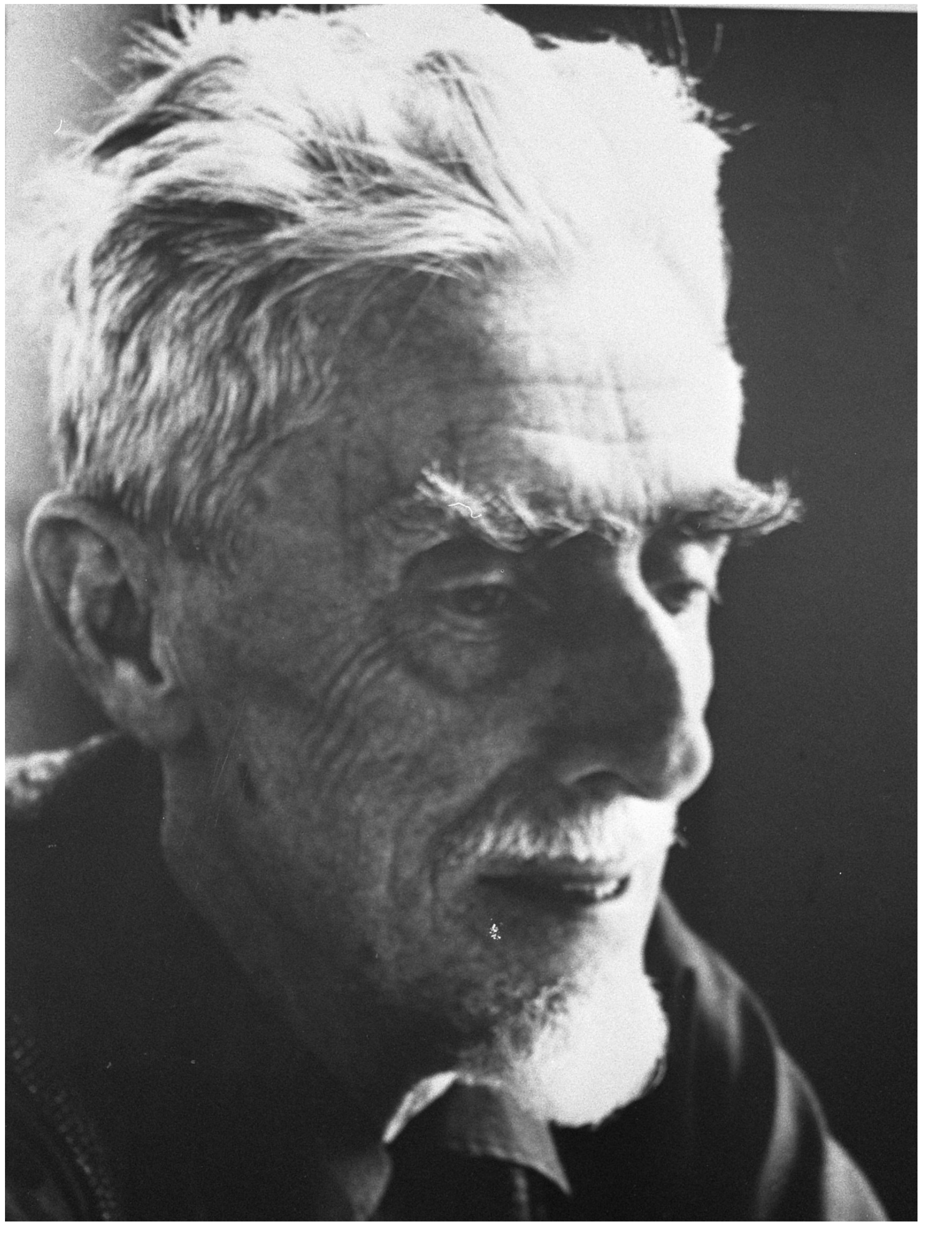


M. C. Escher, Circle Limit II, 1959

M. C. Escher, Circle Limit I, 1958

"...I worked terribly hard to finally finish that litho, and then with gritted teeth, spent another four days making beautiful prints of that extremely complex circle limit in colors. Each print is a series of twenty printings: five pieces, and each piece four times. All this with the remarkable feeling that this work is a milestone in my development, and that nobody, except myself, will ever realize this." – M.C.Escher in a letter to his son Arthur, 1960

Maurits Cornelis Escher, 1898-1972



M. C. Escher in 1971, a photo by Hans Peters