Lecture 15: Anan'in's theorem

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Hyperbolic polyhedral manifolds (reminder)

DEFINITION: A **polyhedral manifold** of dimension 2 is a piecewise smooth manifold obtained by gluing polygons along edges.

DEFINITION: Let $\{P_i\}$ be a set of polygons on the same hyperbolic plane, and M be a polyhedral manifold obtained by gluing these polygons. Assume that all edges which are glued have the same length, and we glue the edges of the same length. Then M is called **a hyperbolic polyhedral manifold**. We consider M as a metric space, with the path metric induced from P_i.

CLAIM: Let M be a hyperbolic polyhedral manifold. Then for each point $x \in M$ which is not a vertex, x **has a neighbourhood which is isometric to an open set of a hyperbolic plane.**

Proof: For interior points of M this is clear. When x belongs to an edge, it is obtained by gluing two polygons along isometric edges, hence the neighbourhood is locally isometric to the union of the same polygons in $\mathbb{H}^2$ aligned along the edge. ■

Fundamental domains and polygons (reminder)

DEFINITION: Let Γ be a discrete group acting on a manifold M, and $U \subset M$ an open subset with piecewise smooth boundary. Assume that for any non-trivial $\gamma \in \Gamma$ one has $U \cap \gamma(U) = \emptyset$ and $\Gamma \cdot \overline{U} = M$, where $\overline{U}$ is closure of U. Then $\overline{U}$ is called **a fundamental domain** of the action of Γ.

THEOREM 1: Let Γ be a discrete group acting on a hyperbolic plane $\mathbb{H}^2$ by oriented isometries. **Then Γ has a polyhedral fundamental domain P.** If, moreover, $\mathbb{H}^2/\Gamma$ has finite volume, ∂P has at most finitely many points on the absolute Abs.

Proof. Step 1: Clearly, $\operatorname{Vol}(P) = \operatorname{Vol}(\mathbb{H}^2/\Gamma)$. This takes care of the last assertion, because polygons with infinitely many points on Abs have infinite volume.
Step 2: Let $\Gamma \cdot x$ be an orbit of Γ in $\mathbb{H}^2$. The Voronoi partition gives a polygonal fundamental domain P for the Γ-action. Clearly, $\mathbb{H}^2$ is tiled by the Voronoi cells, which are all isometric polygons. However, **the cells are not necessarily fundamental domains:** each cell P can be possibly stabilized by a subgroup $\Gamma_P \subset \Gamma$. Clearly, each cell P contains a unique point $x_1 \in \Gamma \cdot x$, hence Γ_P fixes x_1. The stabilizer of x_1 in $SO^+(1,2)$ is a circle, and Γ_P is discrete, hence it is a cyclic group, and P is a polygon admitting an action of a cyclic group. Cutting P onto angles which are freely rotated by Γ_P, we obtain a polygonal fundamental domain. ■

Group quotients and polyhedral hyperbolic manifolds (reminder)

THEOREM: Let $\Gamma \subset SO^+(1,2)$ be a discrete subgroup, and $\mathbb{H}^2/\Gamma$ the quotient. **Then $\mathbb{H}^2/\Gamma$ is isometric to a polyhedral hyperbolic manifold.**

Proof. Step 1: First, **we prove that the quotient $\mathbb{H}^2/\Gamma$ is a manifold.** Indeed, outside of the set of fixed points, the action of Γ is properly discontinuous, and the quotient is smooth. For each fixed point p, it contains a neighbourhood where $\mathrm{St}_\Gamma(p)$ acts as a finite order rotation group $\mathbb{Z}/n\mathbb{Z}$ on a disk, and the quotient is also a disk.

Step 2: By Theorem 1, Γ-action **admits a polygonal fundamental domain D.** Then $\mathbb{H}^2/\Gamma$ is obtained by gluing isometric edges of D; since it is a manifold (Step 1), it is a hyperbolic polyhedral manifold. ■

Fundamental domains and tilings

Earlier, we went from fundamental domains to tilings. **The converse direction is also possible.**

DEFINITION: Bounded polygon in $\mathbb{H}^2$ is a polygon P such that $\overline{P}$ has no points in Abs, or, equivalently, such that the closure of P in $\mathbb{H}^2$ is compact.

CLAIM: Let T be a semi-regular tiling of $\mathbb{H}^2$ by bounded polygons. Then the group Γ of isometries of $\mathbb{H}^2$ preserving T **acts on $\mathbb{H}^2$ with a fundamental domain which is a bounded polygon.**

Proof: Let $\Gamma \cdot x$ be an orbit of $x \in \mathbb{H}^2$, and V_x the corresponding Voronoi domain. It would suffice to show that the closure of V_x is compact. Let $B_x(R) \subset \mathbb{H}^2$ be a a disc of radius R with center in x which contains a representative of each Γ-orbit on the tiles of T. There are finitely many orbits, and all tiles are compact, hence such a disk always exists. Then for every $y \in \mathbb{H}^2$, there exists $\gamma \in \Gamma$ such that $\gamma(y) \in B_x(R)$. Then $d(y, \gamma^{-1}(x)) \leqslant R$, hence either $y \in B_x(R)$ or $y \notin V_x$. We proved that $V_x \subset B_x(R)$, hence the Voronoi polygon is compact. ■

COROLLARY: Let Γ be a group of isometries of a semi-regular tiling. Then the quotient $\mathbb{H}^2/\Gamma$ **is a compact polyhedral hyperbolic manifold,** hence **is a compact Riemann surface.** ■

Hyperbolic polyhedral manifolds: interior angles of vertices (reminder)

DEFINITION: Let $v \in M$ be a vertex in a hyperbolic polyhedral manifold. **Interior angle** of v in M is sum of the adjacent angles of all polygons adjacent to v.

EXAMPLE 1: Let $M \longrightarrow \Delta$ be a ramified n-tuple cover of the Poincaré disk, given by solutions of $y^n = x$. We can lift split Δ to polygons and lift the hyperbolic metric to M, obtaining M as a union of n times as many polygons glued along the same edges. **Then the interior angle of the ramification point is $2\pi n$.**

EXAMPLE 2: Let $\Delta \longrightarrow M$ be a ramified n-tuple cover, obtained as a quotient $M = \Delta/G$, where $G = \mathbb{Z}/n\mathbb{Z}$. Split Δ onto fundamental domains of G, shaped like angles adjacent to 0. Then the quotient Δ/G gives an angle with its opposite sides glued. **It is a hyperbolic polyhedral manifold with interior angle $\frac{2\pi}{n}$ at its ramification point.**

EXAMPLE 3: Let D be a diameter bisecting a disk Δ, and passing through the origin 0 and $P \subset \Delta$ one of the halves. The (unique) edge of P is split onto two half-geodesics E_+ and E_- by the origin. **Gluing E_+ and E_-, we obtain a hyperbolic polyhedral manifold with a single vertex and the interior angle π.**

Sphere with n points of angle π (reminder)

EXAMPLE: Let P be a bounded polygon in $\mathbb{H}^2$, α the sum of its angles, and a_i, $i = 1, ..., n$ median points on its edges E_i. Each a_i splits E_i in two equal intervals. We glue them as in Example 3, and glue all vertices of P together. **This gives a sphere M with hyperbolic polyhedral metric,** one vertex ν with angle α (obtained by gluing all vertices of P together) and n vertices with angle π corresponding to $a_i \in E_i$.

REMARK: Assume that $\alpha = 2\pi$, that is, M is isometric to a hyperbolic disk in a neighbourhood of ν. We equip M with a complex structure compatible with the hyperbolic metric outside of its singularities. A neighbourhood of each singularity is isometrically identified with a neighbourhood of 0 in Δ/G, where $G = \mathbb{Z}/2\mathbb{Z}$. We put a complex structure on Δ/G as in Example 2. **This puts a structure of a complex manifold on M.**

THEOREM: (Alexandre Anan'in)
Let M be the hyperbolic polyhedral manifold obtained from the polygon P with n vertices as above. Assume that n is even, and $\alpha = 2\pi$. **Then M admits a double cover M_1, ramified at all a_i, which is locally isometric to $\mathbb{H}^2$.**

Proof: later today.

REMARK: Clearly, **M_1 is hyperelliptic.**

The proof of Anan'in's Theorem

EXAMPLE: Let P be a bounded convex polygon in $\mathbb{H}^2$, α the sum of its angles, and a_i, $i = 1, ..., n$ the median points on its edges E_i. Each a_i splits E_i in two equal intervals. We glue them as in Example 3, and glue all vertices of P together. **This gives a sphere M with hyperbolic polyhedral metric,** one vertex ν with angle α (obtained by gluing all vertices of P together) and n vertices with angle π corresponding to $a_i \in E_i$.

REMARK: Assume that $\alpha = 2\pi$, that is, M is isometric to a hyperbolic sphere around ν. We equip M with a complex structure compatible with the hyperbolic metric outside of its singularities. A neighbourhood of each singularity is isometrically identified with a neighbourhood of 0 in Δ/G, where $G = \mathbb{Z}/2\mathbb{Z}$.

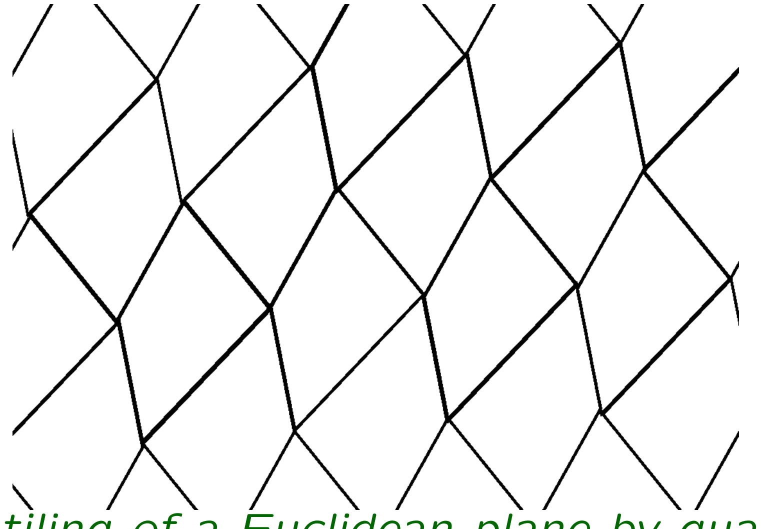
THEOREM: (Alexandre Anan'in)
Let M be the hyperbolic polyhedral manifold obtained from the polyhedron P as above. Assume that n is even, and $\alpha = 2\pi$. **Then M admits a double cover M_1, ramified at all a_i, which is locally isometric to $\mathbb{H}^2$.**

Strategy of the proof: We tile the hyperbolic plane $\mathbb{H}^2$ by copies of P. We show that the group Γ of oriented isometries of this tiling has a subgroup Γ_2 of index 2, freely acting on $\mathbb{H}^2$, such that $\mathbb{H}^2/\Gamma = M$, and $\mathbb{H}^2/\Gamma_2$ is its ramified covering with ramification in $a_1, ..., a_n$.

Alexandre (Sasha) Anan'in, 1952-2021

The proof of Anan'in Theorem (2)

Proof. Step 1: Fix an isometric embedding $P \hookrightarrow \mathbb{H}^2$. Let Γ be the group of isometries generated by central symmetries (that is, rotations with angle π) around $a_1, ..., a_n$. Then the images of P tile $\mathbb{H}^2$ in such a way that P is a fundamental domain of Γ. Indeed, $\Gamma \cdot P$ covers the whole $\mathbb{H}$ (it is open and closed). However, rotating around the edge adjacent to a given vertex of P, we go through a full circle after adding all interior angles one by one. Since the sum of interior angles of P is 2π, we arrive back to P, hence **the images of P intersect with P only on the edge**. We proved that P is a fundamental domain of Γ, which is the isometry group of the tiling of $\mathbb{H}^2$ by copies of P.



Anan'in tiling of a Euclidean plane by quadrangles

The proof of Anan'in Theorem (3)

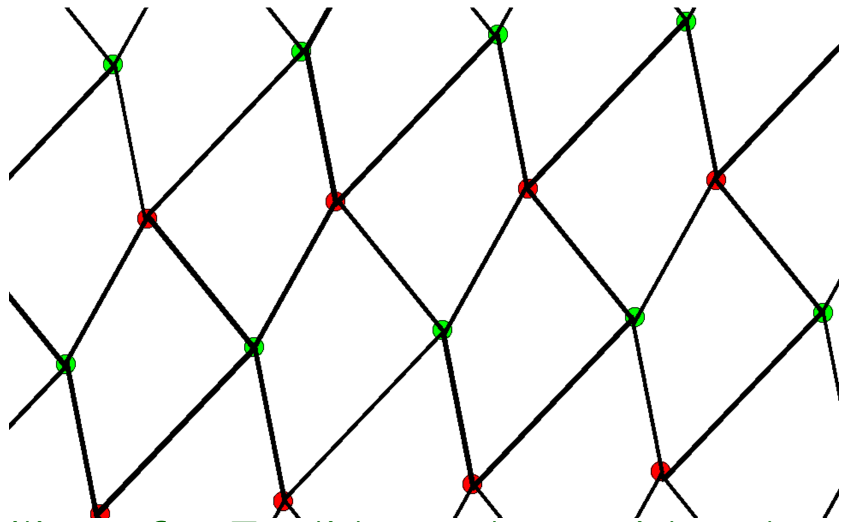
THEOREM: (Alexandre Anan'in)
Let M be the hyperbolic polyhedral manifold obtained from the polyhedron P as above. Assume that n is even, and $\alpha = 2\pi$. **Then M admits a double cover M_1, ramified at all a_i, which is locally isometric to $\mathbb{H}^2$.**

Proof. Step 1: Fix an isometric embedding $P \hookrightarrow \mathbb{H}^2$. Let Γ be the group of isometries generated by central symmetries around $a_1, ..., a_n$. **Then the images of P tile $\mathbb{H}^2$ in such a way that P is a fundamental domain of Γ.**

Step 2: Now we prove that $\mathbb{H}^2/\Gamma = M$. Indeed, Γ acts freely the set of tiles of our tiling, and its action is non-free only in $a_1, ..., a_n$, which are fixed points of appropriate central symmetries. These central symmetries identify two opposite halves of each edge, hence $\mathbb{H}^2/\Gamma$ is obtained by gluing half of each edge of P with the opposite half.

The proof of Anan'in Theorem (4)

Step 3: It remains to construct an index 2 subgroup $\Gamma_2 \subset \Gamma$ freely acting on $\mathbb{H}^2$. We color the vertices of the tiling constructed above in colors red and green in such a way that connected vertices have different colors. This is possible if P has even number of vertices.



Anan'in tiling of a Euclidean plane with colored vertices

The central symmetries τ_j generating Γ exchange red and green vertices. Let $\Gamma_2 \subset \Gamma$ be a subgroup generated by products of even number of τ_j. Clearly, Γ_2 is a subgroup of all elements $\gamma \in \Gamma$ preserving colors of the vertices. Any element of Γ has at most 1 fixed point in the middle of an edge of a tile, **hence Γ_2 acts on $\mathbb{H}^2$ freely.** ■