Lecture 16: Locally constant sheaves

Misha Verbitsky

IMPA, sala 236, 17:00

May 13, 2024

Categories

DEFINITION: A category $\mathcal{C}$ is a collection of data called "objects" and "morphisms between objects" which satisfies the axioms below.

DATA.

Objects: A class $\mathcal{Ob}(\mathcal{C})$ of **objects** of $\mathcal{C}$.

Morphisms: For each $X, Y \in \mathcal{Ob}(\mathcal{C})$, one has a set $\mathcal{Mor}(X, Y)$ of **morphisms from X to Y**.

Composition of morphisms: For each $\varphi \in \mathcal{Mor}(X, Y), \psi \in \mathcal{Mor}(Y, Z)$ there exists **the composition** $\varphi \circ \psi \in \mathcal{Mor}(X, Z)$

Identity morphism: For each $A \in \mathcal{Ob}(\mathcal{C})$ there exists a morphism $\operatorname{Id}_A \in \mathcal{Mor}(A, A)$.

AXIOMS.

Associativity of composition: $\varphi_1 \circ (\varphi_2 \circ \varphi_3) = (\varphi_1 \circ \varphi_2) \circ \varphi_3$.

Properties of identity morphism: For each $\varphi \in \mathcal{Mor}(X, Y)$, one has $\operatorname{Id}_x \circ \varphi = \varphi = \varphi \circ \operatorname{Id}_Y$

Categories (2)

DEFINITION: Let $X, Y \in \mathcal{O}b(\mathcal{C})$ – objects of $\mathcal{C}$. A morphism $\varphi \in \mathcal{M}or(X,Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{M}or(Y,X)$ such that $\varphi \circ \psi = \operatorname{Id}_X$ and $\psi \circ \varphi = \operatorname{Id}_Y$. In this case, the objects X and Y are called **isomorphic**.

Examples of categories:

Category of sets: its morphisms are arbitrary maps.

Category of vector spaces: its morphisms are linear maps.

Categories of rings, groups, fields: morphisms are homomorphisms.

Category of topological spaces: morphisms are continuous maps.

Category of smooth manifolds: morphisms are smooth maps.

Functors

DEFINITION: Let $\mathcal{C}_1, \mathcal{C}_2$ be two categories. A **covariant functor** from $\mathcal{C}_1$ to $\mathcal{C}_2$ is the following set of data.

1. **A map** $F: \mathcal{O}b(\mathcal{C}_1) \longrightarrow \mathcal{O}b(\mathcal{C}_2)$.

2. **A map** $F: \mathcal{M}or(X,Y) \longrightarrow \mathcal{M}or(F(X),F(Y))$ **defined for any pair of objects** $X, Y \in \mathcal{O}b(\mathcal{C}_1)$**.**

These data define a functor if they are **compatible with compositions,** that is, satisfy $F(\varphi) \circ F(\psi) = F(\varphi \circ \psi)$ for any $\varphi \in \mathcal{M}or(X,Y)$ and $\psi \in \mathcal{M}or(Y,Z)$, and **map identity morphism to identity** morphism.

Example of functors

A “natural operation” on mathematical objects is usually a functor. Examples:

1. A map $X \longrightarrow 2^X$ from the set X to the set of all subsets of X is a functor from the category $\mathcal{S}ets$ of sets to itself.

2. A map $M \longrightarrow M^2$ mapping a topological space to its product with itself is a functor on topological spaces.

3. A map $V \longrightarrow V \oplus V$ is a functor on vector spaces; same for a map $V \longrightarrow V \otimes V$ or $V \longrightarrow (V \oplus V) \otimes V$.

4. **Identity functor** from any category to itself.

5. A map from topological spaces to $\mathcal{S}ets$, putting a topological space to the set of its connected components.

EXERCISE: Prove that it is a functor.

Equivalence of functors

DEFINITION: Let $X, Y \in \mathcal{O}b(\mathcal{C})$ be objects of a category $\mathcal{C}$. A mprphism $\varphi \in \mathcal{M}or(X,Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{M}or(Y,X)$ such that $\varphi \circ \psi = \operatorname{Id}_X$ and $\psi \circ \varphi = \operatorname{Id}_Y$. In this case X and Y are called **isomorphic**.

DEFINITION: Two functors $F, G: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ are called **equivalent** if for any $X \in \mathcal{O}b(\mathcal{C}_1)$ we are given an isomorphism $\Psi_X: F(X) \longrightarrow G(X)$, in such a way that for any $\varphi \in \mathcal{M}or(X,Y)$, one has $F(\varphi) \circ \Psi_Y = \Psi_X \circ G(\varphi)$.

REMARK: Such commutation relations are usually expressed by **commutative diagrams**. For example, the condition $F(\varphi) \circ \Psi_Y = \Psi_X \circ G(\varphi)$ is expressed by a commutative diagram

$$\begin{tikzcd}
F(X) \arrow[r, "F(\varphi)"] \arrow[d, "\Psi_X"'] & F(Y) \arrow[d, "\Psi_Y"] \\
G(X) \arrow[r, "G(\varphi)"] & G(Y)
\end{tikzcd}$$

Equivalence of categories

DEFINITION: A functor $F: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is called **equivalence of categories** if there exists a functor $G: \mathcal{C}_2 \longrightarrow \mathcal{C}_1$ such that the compositions $G \circ F$ and $G \circ F$ are equivaleent to the identity functors $\operatorname{Id}_{\mathcal{C}_1}$, $\operatorname{Id}_{\mathcal{C}_2}$.

REMARK: It is possible to show that this is equivalent to the following conditions: **F defines a bijection on the set of isomorphism classes of objects of $\mathcal{C}_1$ and $\mathcal{C}_2$, and a bijection**

$$\mathcal{M}or(X,Y) \longrightarrow \mathcal{M}or(F(X), F(Y)).$$

for each $X, Y \in \mathcal{O}b(\mathcal{C}_1)$.

REMARK: From the point of view of category theory, **equivalent categories are two instances of the same category** (even if the cardinality of corresponding sets of objects is different).

Sheaves

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

REMARK: The definition of a sheaf below **is a more abstract version of the notion of "sheaf of functions"** defined previously.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces (or abelian groups) $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $R_{UW} = R_{UV} \circ R_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and the restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**
2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Sheaves and exact sequences

DEFINITION: A sequence $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ of homomorphisms of abelian groups or vector spaces is called **exact** if the image of each map is the kernel of the next one.

CLAIM: A presheaf $\mathcal{F}$ is a sheaf if and only if for every cover $\{U_i\}$ of an open subset $U \subset M$, **the sequence of restriction maps**

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, with $\eta \in \mathcal{F}(U_i)$ mapped to

$$\eta\Big|_{U_i \cap U_j} - \eta\Big|_{U_j \cap U_i}.$$

Locally constant sheaves

DEFINITION: Let $\mathcal{F}$ be a sheaf on M which takes a connected non-empty open subset $U \subset M$ to a vector space or abelian group $\mathbb{V}$. Extend $\mathcal{F}$ to all open sets using the gluing axiom. Then $\mathcal{F}$ is called **the constant sheaf**, denoted $\mathbb{V}_M$.

EXERCISE: Prove that **the constant sheaf $\mathbb{V}_M$ exists, and is unique up to isomorphism.**

EXERCISE: Let W be an open set in M, and S_W its set of connected components. Prove that $\mathbb{V}_M(W) = \mathbb{V}^{|S_W|}$.

DEFINITION: A **locally constant sheaf** is a sheaf which is locally isomorphic to a constant sheaf.

EXAMPLE: Let $\pi: \ M' \longrightarrow M$ be a covering. Given $U \subset M$, let S_U be the set of connected components of $\pi^{-1}(U)$, and set $\mathcal{F}(U) = \mathbb{V}^{|S_W|}$. We are going to define the restriction map r as follows. For an open subset $W \subset U$, consider the map $S_W \longrightarrow S_U$ induced by the natural embedding $\pi^{-1}(W) \overset{j}{\hookrightarrow} \pi^{-1}(U)$. For each direct sum component $\mathbb{V}_u \subset \mathbb{V}^{|S_U|}$ corresponding to $u \in \operatorname{im} j$, let $r_u: \ \mathbb{V}_u \longrightarrow \mathbb{V}_{j(u)}$ be identity. For a component $\mathbb{V}_u \subset \mathbb{V}^{|S_U|}$ corresponding to $u \notin \operatorname{im} j$, we set $r_u = 0$. Then $r := \bigoplus_{u \in S_U} r_u: \ \bigoplus_{u \in S_U} \mathbb{V} \longrightarrow \bigoplus_{w \in S_W} \mathbb{V}$. **This defines a locally constant sheaf on M (prove it).**

Étalé space of a sheaf

DEFINITION: Let $\mathcal{F}$ be a sheaf on M, and $U, V \supset x$ be two open set containing $x \in M$. Two sections $f \in \mathcal{F}(U)$, $g \in \mathcal{F}(V)$ are called **equivalent in x** if there exists an open set $W \ni x$ such that $W \subset U \cap V$ and $f|_W = g|_W$. **A germ of a sheaf $\mathcal{F}$ in x** is a class of equivalence of sections of $\mathcal{F}$ in all open sets $U \ni x$ under this equivalence relation. **The stalk** of a sheaf $\mathcal{F}$ in x is the space $\mathcal{F}_x$ of all germs in x.

DEFINITION: Let $E(\mathcal{F})$ be the set of all stalks of a sheaf $\mathcal{F}$ in all points $x \in M$. A germ $f \in \mathcal{F}_m$ is called **a limit of a sequence of germs $f_i \in \mathcal{F}_{m_i}$** if $\lim_i m_i = m$ and there exists a section $\tilde{f}$ of $\mathcal{F}$ over $U \ni x$ such that almost all f_i are germs of $\tilde{f}$. The **étalé topology** on $E(\mathcal{F})$ is defined as follows: a subset $K \subset E(\mathcal{F})$ is **closed in étalé topology** if it contains all its limit points.

REMARK: Usually $E(\mathcal{F})$ **is non-Hausdorff**.

Étalé space of a constant sheaf

CLAIM: Let $\mathcal{F} = \mathbb{V}_M$ be a constant sheaf on a manifold, and $x \in M$ a connected subset. **Then the space of germs of $\mathcal{F}$ in x is equal to $\mathbb{V}$.**

Proof: Since $\mathcal{F}$ is constant, the set of its sections on any connected open set is equal to $\mathbb{V}$. This gives a natural map $r_x := \mathcal{F}(U) \longrightarrow \mathbb{V}$: we restrict $f \in \mathcal{F}(U)$ to a connected component U_1 of U containing x, and obtain an element of $\mathbb{V}$. **Clearly, two sections f, g are equivalent in K if and only if $r_x(f) = r_x(g)$.** This identifies $\mathbb{V}$ with the set of equivalence classes of sections in x. ■

Corollary 1: Let $\mathcal{F} = \mathbb{V}_M$ be a constant sheaf on a manifold. **Then the étalé space $E(\mathcal{F})$ of $\mathcal{F}$ is identified with $\mathbb{V}$ disconnected copies of M.**

Proof: Indeed, a sequence $f_i \in \mathcal{F}_{m_i}$ converges to f if $\lim_i m_i = m$ and $r_{m_i}(f_i) = r_m(f)$ for almost all i. ■

Local systems

DEFINITION: Category of coverings of M is category $\mathcal{C}$ with $\mathcal{O}b(\mathcal{C})$ all coverings and morphisms continuous maps of coverings compatible with projections to M.

DEFINITION: Let $\pi_1 : M_1 \longrightarrow M$, $\pi_2 : M_2 \longrightarrow M$ be continuous maps. **Fibered product** $M_1 \times_M M_2$ is the subset of $M_1 \times M_2$ defined as $M_1 \times_M M_2 := \{(x, y) \in M_1 \times M_2 \mid \pi_1(x) = \pi_2(y)\}$, with induced topology.

EXERCISE: Prove that **a fibered product of coverings is a covering**.

DEFINITION: An abelian group structure on a covering $\pi_1 : M_1 \longrightarrow M$ is a morphism of coverings $\mu : M_1 \times_M M_1 \longrightarrow M_1$ together with a morphism $e : M \longrightarrow M_1$ from a trivial covering to M_1 and $\in \operatorname{Hom}_M(M_1)$ such that μ defines an additive structure of an abelian group on the set $\pi_1^{-1}(x)$ for each $x \in M$, with $e(x)$ a unit in this group and a the inverse.

REMARK: If, in addition, we have a group homomorphism $\mathbb{R}^* \longrightarrow \operatorname{Aut}_M(M_1, M_1)$ which equips each $\pi_1^{-1}(x)$ with a structure of a vector space, we obtain **a structure of a vector space on a covering**.

DEFINITION: A local system is a covering with a structure of an abelian group or a vector space.

Étalé space of a locally constant sheaf

THEOREM: Let $\mathcal{F} = \mathbb{V}_M$ be a locally constant sheaf on a manifold. **Then its étalé space $E(\mathcal{F})$ is a covering of M.**

Proof: Immediately follows from Corollary 1. ■

THEOREM: Category of locally constant sheaves is equivalent to the category of local systems.

Proof: Let $\mathcal{F}$ be a locally constant sheaf, and $E(\mathcal{F})$ its etale space. Then $E(\mathcal{F})$ is a covering of M. The structure of vector space on germs defines the structure of vector space on $E(\mathcal{F})$. **This gives a functor from locally constant sheaves to local systems.**

Conversely, let $\pi: M_1 \longrightarrow M$ be a local system, and $\mathcal{F}(U)$ be the space of the sections of $\pi^{-1}(U) \xrightarrow{\pi} U$. Then $\mathcal{F}(U)$ is a vector space. The correspondence $U \longrightarrow \mathcal{F}(U)$ gives a sheaf, which is clearly locally constant. ■