Lecture 17: Riemann-Hilbert correspondence

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IMPA, sala 236, 17:00

May 15, 2024

Étalé space of a sheaf (reminder)

DEFINITION: Let $\mathcal{F}$ be a sheaf on M, and $U, V \supset x$ be two open set containing $x \in M$. Two sections $f \in \mathcal{F}(U)$, $g \in \mathcal{F}(V)$ are called **equivalent in** x if there exists an open set $W \ni x$ such that $W \subset U \cap V$ and $f|_W = g|_W$. **A germ of a sheaf $\mathcal{F}$ in x** is a class of equivalence of sections of $\mathcal{F}$ in all open sets $U \ni x$ under this equivalence relation. **The stalk** of a sheaf $\mathcal{F}$ in x is the space $\mathcal{F}_x$ of all germs in x.

DEFINITION: Let $E(\mathcal{F})$ be the set of all stalks of a sheaf $\mathcal{F}$ in all points $x \in M$. A germ $f \in \mathcal{F}_m$ is called **a limit of a sequence of germs** $f_i \in \mathcal{F}_{m_i}$ if $\lim_i m_i = m$ and there exists a section $\tilde{f}$ of $\mathcal{F}$ over $U \ni x$ such that almost all f_i are germs of $\tilde{f}$. The **étalé topology** on $E(\mathcal{F})$ is defined as follows: a subset $K \subset E(\mathcal{F})$ is **closed in étalé topology** if it contains all its limit points.

REMARK: Usually $E(\mathcal{F})$ **is non-Hausdorff**.

Étalé space of a constant sheaf (reminder)

CLAIM: Let $\mathcal{F} = \mathbb{V}_M$ be a constant sheaf on a manifold, and $x \in M$ a connected subset. **Then the space of germs of $\mathcal{F}$ in x is equal to $\mathbb{V}$.**

Proof: Since $\mathcal{F}$ is constant, the set of its sections on any connected open set is equal to $\mathbb{V}$. This gives a natural map $r_x := \mathcal{F}(U) \longrightarrow \mathbb{V}$: we restrict $f \in \mathcal{F}(U)$ to a connected component U_1 of U containing x, and obtain an element of $\mathbb{V}$. **Clearly, two sections f, g are equivalent in K if and only if $r_x(f) = r_x(g)$.** This identifies $\mathbb{V}$ with the set of equivalence classes of sections in x. ■

Corollary 1: Let $\mathcal{F} = \mathbb{V}_M$ be a constant sheaf on a manifold. **Then the étalé space $E(\mathcal{F})$ of $\mathcal{F}$ is identified with $\mathbb{V}$ disconnected copies of M.**

Proof: Indeed, a sequence $f_i \in \mathcal{F}_{m_i}$ converges to f if $\lim_i m_i = m$ and $r_{m_i}(f_i) = r_m(f)$ for almost all i. ■

Local systems

DEFINITION: **Category of coverings** of M is category $\mathcal{C}$ with $\mathcal{O}b(\mathcal{C})$ all coverings and morphisms continuous maps of coverings compatible with projections to M.

DEFINITION: Let $\pi_1 : M_1 \longrightarrow M$, $\pi_2 : M_2 \longrightarrow M$ be continuous maps. **Fibered product** $M_1 \times_M M_2$ is the subset of $M_1 \times M_2$ defined as $M_1 \times_M M_2 := \{(x,y) \in M_1 \times M_2 \mid \pi_1(x) = \pi_2(y)\}$, with induced topology.

EXERCISE: Prove that **a fibered product of coverings is a covering**.

DEFINITION: **An abelian group structure on a covering $\pi_1 : M_1 \longrightarrow M$** is a morphism of coverings $\mu : M_1 \times_M M_1 \longrightarrow M_1$ together with a morphism $e : M \longrightarrow M_1$ from a trivial covering to M_1 and $\in \operatorname{Hom}_M(M_1)$ such that μ defines an additive structure of an abelian group on the set $\pi_1^{-1}(x)$ for each $x \in M$, with $e(x)$ a unit in this group and a the inverse.

REMARK: If, in addition, we have a group homomorphism $\mathbb{R}^* \longrightarrow \operatorname{Aut}_M(M_1, M_1)$ which equips each $\pi_1^{-1}(x)$ with a structure of a vector space, we obtain **a structure of a vector space on a covering**.

DEFINITION: **A local system** is a covering with a structure of an abelian group or a vector space.

Étalé space of a locally constant sheaf

THEOREM: Let $\mathcal{F} = \mathbb{V}_M$ be a locally constant sheaf on a manifold. **Then its étalé space $E(\mathcal{F})$ is a covering of M.**

Proof: Immediately follows from Corollary 1. ■

THEOREM: **Category of locally constant sheaves is equivalent to the category of local systems.**

Proof: Let $\mathcal{F}$ be a locally constant sheaf, and $E(\mathcal{F})$ its etale space. Then $E(\mathcal{F})$ is a covering of M. The structure of vector space on germs defines the structure of vector space on $E(\mathcal{F})$. **This gives a functor from locally constant sheaves to local systems.**

Conversely, let $\pi : M_1 \longrightarrow M$ be a local system, and $\mathcal{F}(U)$ be the space of the sections of $\pi^{-1}(U) \xrightarrow{\pi} U$. Then $\mathcal{F}(U)$ is a vector space. The correspondence $U \longrightarrow \mathcal{F}(U)$ gives a sheaf, which is clearly locally constant. ■

Connections

Notation: Let M be a smooth manifold, TM its tangent bundle, $\Lambda^i M$ the bundle of differential i-forms, $C^\infty M$ the smooth functions. **The space of sections of a bundle B is denoted by B.**

DEFINITION: A **connection** on a vector bundle B is an operator $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, where $f \longrightarrow df$ is de Rham differential. When X is a vector field, we denote by $\nabla_X(b) \in B$ the term $\langle \nabla(b), X \rangle$.

REMARK: When $M = [0, a]$ is an interval, any bunlde B on M is trivial. Let $b_1, ..., b_n$ be a basis in B. Then ∇ can be written as

$$\nabla_{d/dt}\left(\sum f_i b_i\right) = \sum_i \frac{df_i}{dt} b_i + \sum f_i \nabla_{d/dt} b_i$$

with the last term linear on f. Therefore, the equation $\nabla_{d/dt}(b) = 0$ is a first order ODE, and **it has a unique solution for any initial value $b_0 = b\big|_{\{0\}}$.**

Curvature

DEFINITION: Let $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a vector bundle B. We extend ∇ to an operator

$$V \xrightarrow{\nabla} \Lambda^1(M) \otimes B \xrightarrow{\nabla} \Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B \xrightarrow{\nabla} ...$$

using the Leibnitz identity $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde\eta}\eta \wedge \nabla b$. Then the operator $\nabla^2: B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇.

REMARK: $\nabla^2(fb) = d^2 fb + df \wedge \nabla b - df \wedge \nabla b + f\nabla^2 b$, hence **the curvature is a $C^\infty M$-linear operator. We shall consider the curvature B as a 2-form with values in End B.** Then $\nabla^2 := \Theta_B \in \Lambda^2 M \otimes \operatorname{End} B$, where an End($B$)-valued form acts on $\Lambda^* M \otimes B$ as above.

DEFINITION: A connection is **flat** if its curvature vanishes.

Riemann-Hilbert correspondence

THEOREM: Let M be a connected manifold, $\mathcal{C}_1$ the category of representations of $\pi_1(M)$, and $\mathcal{C}_2$ the category of local systems. **Then the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are naturally equivalent.**

Proof: The category of sets with an action of $\pi_1(M)$ is equivalent to the category of coverings (assignment 7). Local systems are group objects in the category of coverings, and representations of $\pi_1(M)$ are group objects in the category of sets with an action of $\pi_1(M)$. ■

THEOREM: The categories $\mathcal{C}_1$ and $\mathcal{C}_2$ **are naturally equivalent to the category of vector bundles on M equipped with flat connection.**

Proof: We shall do it soon.

Vector fields and derivations

DEFINITION: Let M be a smooth manifold, and $X: C^\infty M \longrightarrow C^\infty M$ an operator. We call X **a vector field** if in each coordinate system $x_1, ..., x_n$ on M, the map X can be written as $X(f) = \sum \alpha_i \frac{df}{dx_i}$.

DEFINITION: A map X from a ring to itself is called **a derivation** it it satisfies **the Leibnitz rule**: $X(fg) = gX(f) + fX(g)$. Further on, we shall mostly consider derivations $X: C^\infty M \longrightarrow C^\infty M$. Such derivations are tacitly **assumed to be $\mathbb{R}$-linear.**

DEFINITION: **Support** Supp(f) of a continuous function f is the closure of the set of all points where it is not equal to 0. An operator $X: C^\infty M \longrightarrow C^\infty M$ is **local** if it maps a function with support in K to a function with support in K, for each $K \subset M$.

REMARK: Clearly, **all vector fields are local derivations.**

Derivations are local

THEOREM: Any derivation $X: C^\infty M \longrightarrow C^\infty M$ is local.

Proof. Step 1: Let $K, L \subset M$ be non-intersecting closed sets, f be a function with support in K, and g a function satisfying $g|_L = 1$ and $\operatorname{Supp}(g) \cap K = \emptyset$. Then $0 = X(fg) = gX(f) + fX(g)$. Restricted to L, **this gives $X(f)|_L = 0$,** because $g|_L = 1$ and $f|_L = 0$.

Step 2: Let us prove now that $\operatorname{Supp}(X(f)) \subset K = \operatorname{Supp}(f)$. Let $x \notin K$. We need to show that $x \notin \operatorname{Supp}(X(f))$.

Choosing an appropriate coordinate system on $M \backslash K$, we find a function g such that $\operatorname{Supp}(g) \subset M \backslash K$ and $g = 1$ in a neighbourhood V of x. Then $X(f)|_V = 0$ by Step 1. **This implies that $x \notin \operatorname{Supp}(X(f))$.** ■

Hadamard's Lemma

LEMMA: (Hadamard's Lemma)
Let f be a smooth function on $\mathbb{R}^n$, and x_i the coordinate functions. **Then $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$, for some smooth $g_i \in C^\infty \mathbb{R}^n$.**

Proof: Let $t \in \mathbb{R}^n$. Consider a function $h(t) \in C^\infty \mathbb{R}^n$, $h(t) = f(tx)$. Using the chain rule, we get $\frac{dh}{dt} = \sum \frac{d}{dx_i} f(tx) x_i$, obtaining

$$f(x) - f(0) = \int_0^1 \frac{dh}{dt} dt = \sum_i x_i \int_0^1 \frac{df(tx)}{dx_i}(tx) dt.$$

■

COROLLARY: Let $\mathfrak{m}_0$ be an ideal of all smooth functions on $\mathbb{R}^n$ vanishing in 0. **Then $\mathfrak{m}_0$ is generated by coordinate functions.** ■

COROLLARY: Let f be a smooth function on $\mathbb{R}^n$ satisfying $f(x) = 0$ and $df\Big|_{T_x(M)} = 0$. **Then $f \in \mathfrak{m}_x^2$.**

Proof: $f(x) = \sum_{i=1}^n x_i g_i(x)$, where all g_i vanish in 0. ■

EXERCISE 1: Let $X : C^\infty M \longrightarrow C^\infty M$ be a derivation, and $f \in \mathfrak{m}_x^2$. **Prove that $X(f) \in \mathfrak{m}_x$.**

Polynomial vector fields and derivations

LEMMA 1: Let k be any field, $k[x_1, ..., x_n]$ the ring of polynomials, and $A \supset k[x_1, ..., x_n]$ any ring. Then **any k-linear derivation $X: k[x_1, ..., x_n] \longrightarrow A$ is expressed as $X(F) = \sum \alpha_i \frac{df}{dx_i}$, where $\alpha_i = X(x_i)$.**

Proof: A derivation X on $\mathbb{R}[x_1, ..., x_n]$ is determined by Leibnitz formula and $X(x_1), ..., X(x_n)$. On monomials the Leibnitz formula gives

$$X(x_1^{a_1} x_2^{a_2} ... x_n^{a_n}) = a_1 X(x_1) x_1^{a_1-1} x_2^{a_2} ... x_n^{a_n} + a_2 X(x_2) x_1^{a_1} x_2^{a_2-1} ... x_n^{a_n} + ... \\ + a_n X(x_n) x_1^{a_1} x_2^{a_2-1} ... x_n^{a_n-1}.$$

Therefore, for any polynomial $F \in \mathbb{R}[x_1, ..., x_n]$, we have $X(F) = \sum \alpha_i \frac{dF}{dx_i}$, where $\alpha_i = X(x_i)$. ■

Vector fields and derivations

THEOREM 1: Any derivation $X: C^\infty M \longrightarrow C^\infty M$ is a vector field on M.

Proof. Step 1: Since derivations are local, **it suffices to prove this statement on $\mathbb{R}^n$.**

Step 2: For polynomial functions $F \in \mathbb{R}[x_1, ..., x_n]$, we have $X(F) = \sum \alpha_i \frac{dF}{dx_i}$, where $\alpha_i = X(x_i)$ (Lemma 1). **Therefore, $X\Big|_{\mathbb{R}[x_1,...,x_n]}$ is a vector field.**

Step 3: Given a derivation δ, write $\delta_0 := \sum \alpha_i \frac{df}{dx_i}$, where $\alpha_i = \delta(x_i)$. To finish Theorem 1, it suffices to show that $\delta_1 := \delta - \delta_0 = 0$. **This would follow if we prove that all derivations δ_1 vanishing on all polynomial functions vanish.**

Step 4: By Hadamard's lemma, for each $x \in \mathbb{R}^n$, one has $f \in \mathfrak{m}_x^2$ modulo linear functions. Since δ_1 vanishes on linear functions, one has $\delta_1(C^\infty\mathbb{R}^n) = \delta_1(\mathfrak{m}_x^2)$ for each $x \in \mathbb{R}^n$ (Hadamard's lemma). However, $\delta_1(\mathfrak{m}_x^2) \in \mathfrak{m}_x$ because δ_1 is a derivation (Exercise 1). **Therefore, $\delta_1(f)$ vanishes everywhere for all $f \in C^\infty\mathbb{R}^n$.** ■

Flow of diffeomorphisms

DEFINITION: Let $V: M\times [a,b]\longrightarrow M$ be a smooth map such that for all $t\in [a,b]$ the restriction $V_t := V\big|_{M\times\{t\}}: M\longrightarrow M$ is a diffeomorphism. Then V is called **a flow of diffeomorphisms**.

CLAIM: Let V_t be a flow of diffeomorphisms, $f\in C^\infty M$, and $V_t^*(f)(x) := f(V_t(x))$. Consider the map $\frac{d}{dt}V_t|_{t=c}: C^\infty M\longrightarrow C^\infty M$, with $\frac{d}{dt}V_t|_{t=c}(f) = (V_c^{-1})^*\frac{dV_t}{dt}|_{t=c}f$. **Then $f\longrightarrow (V_t^{-1})^*\frac{d}{dt}V_t^* f$ is a derivation** (that is, a vector field).

Proof: $\frac{d}{dt}V_t^*(fg) = V_t^*(f)\frac{d}{dt}\cdot V_t^* g + \frac{d}{dt}V_t^* f\cdot V_t^*(g)$ by the Leibniz rule, giving

$$(V_t^{-1})^*\frac{d}{dt}V_t^*(fg) = f\cdot (V_t^{-1})^*\frac{d}{dt}V_t^* g + g\cdot (V_t^{-1})^*\frac{d}{dt}V_t^* f.$$

■

DEFINITION: The vector field $\frac{d}{dt}V_t|_{t=c}$ is called **a vector field tangent to a flow of diffeomorphisms V_t at $t=c$.**

DEFINITION: Let v_t be a vector field on M, smoothly depending on the time parameter $t\in [a,b]$, and $V: M\times [a,b]\longrightarrow M$ a flow of diffeomorphisms which satisfies $(V_t^{-1})^*\frac{d}{dt}V_t = v_t$ for each $t\in [a,b]$, and $V_0 = \operatorname{Id}$. Then V_t is called **an exponent of v_t**.

Automorphisms of the ring of functions

REMARK: Each diffeomorphism $\psi: M \longrightarrow M$ induces an automorphism of the ring of smooth functions on M, $f \mapsto \psi^* f$.

THEOREM: Let M be a manifold. Then **any automorphism $\Psi: C^\infty M \longrightarrow C^\infty M$ is induced by a diffeomorphism of M.**

Proof. Step 1: Given a point $x \in M$, denote by I_x **the maximal ideal of x**, that is, the ideal of all functions vanishing in x. On a compact manifold, any maximal ideal is obtained this way. Indeed, if an ideal $I \subset C^\infty M$ has no common zeros, for each $y \in M$ there exists $f_y \in I$ which does not vanish in y. Denote by U_y the open set where $f_y \neq 0$. Then $\{U_y\}$ is an open cover of M. Finding a finite subcover, we obtain a finite number of functions $f_i \in I$ such that $\bigcap_i U_{f_i} = M$. Then **the function $\sum f_i^2 \in I$ is invertible, hence $I = C^\infty M$ is not a maximal ideal.** For non-compact manifolds, **points of M are the same as ideals $I \subset C^\infty M$ such that $C^\infty M/I = \mathbb{R}$ (prove it).**

Step 2: Identifying points and maximal ideals, we obtain a map $\psi: M \longrightarrow M$ induced by Ψ. **It remains to show that this map is a diffeomorphism.**

Automorphisms of the ring of functions (2)

THEOREM: Let M be a compact manifold. Then **any automorphism $\Psi: C^\infty M \longrightarrow C^\infty M$ is induced by a diffeomorphism of M.**

Step 2: Identifying points and maximal ideals, we obtain a map $\psi: M \longrightarrow M$ induced by Ψ. **It remains to show that this map is a diffeomorphism.**

Step 3: All open subsets of M can be obtained as unions of open sets $U_f := f^{-1}(\mathbb{R}\backslash 0)$, where $f \in C^\infty M$ **(prove it)**. However, $f(x) = 0$ if and only if $f \in I_x$. Then U_f can be considered as a set of maximal ideals I_x such that $f \notin I_x$. Since Ψ maps U_f to $U_{\Psi(f)}$, the corresponding map ψ is continuous on M. This implies that ψ **is a homeomorphism.**

Step 4: Finally, Ψ maps coordinate functions on $U \subset M$ to coordinate functions on $\psi^{-1}(U)$, hence this homeomorphism is smooth. ∎

Solutions of ODE (1)

DEFINITION: Let v_t be a vector field on M, smoothly depending on the time parameter $t \in [0,a]$, and $V: M \times [0,a] \longrightarrow M$ a flow of diffeomorphisms which satisfies $(V_t^{-1})^* \frac{d}{dt} V_t = v_t$ for each $t \in [0,a]$, and $V_0 = \operatorname{Id}$. Then V_t is called **an exponent of v_t**.

Theorem 1: Let v_t be a vector field on M, smoothly depending on the time parameter $t \in [0,a]$. Then **the exponent of v_t is unique. It always exists** when v_t has compact support.

Solutions of ODE (2)

Theorem 2: Let v_t be a vector field on M, smoothly depending on the time parameter $t\in [0,a]$. Then **the exponent of v_t is unique. It always exists** when v_t vanish (for all t) outside of a compact set $K\subset M$.

Proof: To construct a flow of diffeomorphisms $V_t = e^{v_t}$ it suffices to find a family of automorphisms $\Psi_t: C^\infty M \longrightarrow C^\infty M$ smoothly depending on $t\in [0,a]$ such that $\Psi_t^{-1}\frac{d}{dt}\Psi_t = v_t$. This is the same as to solve the ordinary differential equation

$$\frac{df_t}{dt} = v_t(f_t) \qquad (*)$$

for any given f_0. Then $\Psi_t(f_0) := f_t$ clearly satisfies $\frac{d}{dt}\Psi_t(f_0) = v_t\Psi_t(f_0)$.

To finish the proof, **we need to show that a solution of (*) exists and is unique,** and **to prove that Ψ_t defined this way is an automorphism, that is, satisfies $\Psi_t(fg) = \Psi_t(f)\Psi_t(g)$.**

Existence and uniqueness of solutions of ODE

THEOREM: Let v_t be a vector field on a manifold M. Consider the differential equation

$$\frac{dx_t}{dt} = v_t(x_t), \qquad (*)$$

where $x_t \in M$, and $t \in [0,a]$. Suppose that v_t has compact support. **Then (*) has a unique solution for each initial value x_0.**

Proof: Existence and uniqueness of solutions of (*) follows from Peano and Picard-Lindelöf theorem. Recall that a function $\mu: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **Lipschitz** if $|\mu(x) - \mu(y)| < C|x-y|$ for all x, y. Let D be an open subset of $\mathbb{R} \times \mathbb{R}^n$, $f \in C^\infty D$, and

$$\frac{df_t}{dt} = v(t, f(t)) \qquad (**)$$

a continuous first-order differential equation defined on D. **(Peano) Then for every initial value f_0 there exists a solution of (**) defined on a small interval $[0,\varepsilon]$.** Moreover **(Picard-Lindelöf) the solution is unique if v is Lipschitz**. Notice that v is Lipschitz on any compact set if it is smooth. Finally, if there are functions $\alpha, \beta: [0,\infty[\longrightarrow [0,\infty[$ such that $|v_t(x)| < \alpha(t)|x| + \beta(t)$, **the solution exists globally for all $t \in [0,\infty[$.** ■

Derivations and automorphisms

To finish Theorem 2, **it would suffice to show that the map $f_0 \xrightarrow{\Psi_t} f_t$ obtained as a solution of $\frac{df_t}{dt} = v_t(f_t)$ is multiplicative:** $\Psi_t(fg) = \Psi_t(f)\Psi_t(g)$. From the definition of Ψ_t it follows

$$\frac{d}{dt}\Psi_t(fg) = v_t(f_t)g_t + f_t v(g_t)$$

and

$$\frac{d}{dt}\Big(\Psi_t(f)\Psi_t(g)\Big) = v_t(f_t)g_t + f_t v(g_t)$$

Therefore, both $\Psi_t(fg)$ and $\Psi_t(f)\Psi_t(g)$ are solition of a differential equation $\frac{d}{dt}(\chi_t) = v_t(\chi_t)$ with the same initial value $\chi_0 = fg$. **They are equal by uniqueness of solutions.**

The same argument proves the following lemma.

LEMMA: Let v, v' be commuting vector fields. **Then the corresponding diffeomorphisms commute.** Moreover, **$V_t(v') = v'$, where V_t is the diffeomorphism flow associated with v.**

Proof: Indeed, **exponents of commuting linear operators commute.** ■