Lecture 18: Frobenius theorem

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Distributions

DEFINITION: Distribution on a manifold is a sub-bundle $B \subset TM$

REMARK: Let $\Pi : TM \longrightarrow TM/B$ be the projection, and $x, y \in B$ some vector fields. Then $[fx, y] = f[x, y] - D_y(f)x$. This implies that $\Pi([x, y])$ **is $C^\infty(M)$-linear as a function of x and y.**

DEFINITION: The map $[B, B] \longrightarrow TM/B$ we have constructed is called **Frobenius bracket** (or **Frobenius form**); it is a skew-symmetric $C^\infty(M)$-linear form on B with values in TM/B.

DEFINITION: A distribution is called **holonomic**, or **involutive**, if its Frobenius form vanishes.

Smooth submersions

DEFINITION: Let $\pi : M \longrightarrow M'$ be a smooth map of manifolds. This map is called **submersion** if at each point of M the differential $D\pi$ is surjective, and **immersion** if it is injective.

CLAIM: Let $\pi : M \longrightarrow M'$ be a submersion. Then each $m \in M$ has a neighbourhood $U \cong V \times W$, where V, W are smooth and **$\pi|_U$ is a projection of $V \times W = U \subset M$ to $W \subset M'$ along V.**

Proof: Follows from the inverse function theorem. ■

THEOREM: ("Ehresmann's fibration theorem")
Let $\pi : M \longrightarrow M'$ be a smooth submersion of compact manifolds. **Prove that π is a locally trivial fibration.**

Proof: Next slide.

DEFINITION: Vertical tangent space $T_\pi M \subset TM$ of a submersion $\pi : M \longrightarrow M'$ is the kernel of $D\pi$.

Ehresmann connections

DEFINITION: Let $\pi : M \longrightarrow Z$ be a smooth submersion, with $T_\pi M$ **the bundle of vertical tangent vectors** (vectors tangent to the fibers of π). An **Ehresmann connection** on π is a sub-bundle $T_{\mathrm{hor}}M \subset TM$ such that $TM = T_{\mathrm{hor}}M \oplus T_\pi M$. The **parallel transport** along the path $\gamma : [0,a] \longrightarrow Z$ associated with the Ehresmann connection is a diffeomorphism

$$V_t : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(t))$$

smoothly depending on $t \in [0,a]$ and satisfying $\frac{dV_t}{dt} \in T_{\mathrm{hor}}M$.

CLAIM: Let $\pi : M \longrightarrow Z$ be a smooth fibration with compact fibers. Then **the parallel transport, associated with the Ehresmann connection, always exists.**

Proof: Follows from existence and uniqueness of solutions of ODEs. ■

Foliations

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \xrightarrow{\pi} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

REMARK: The implication **"$B = T_\pi M$" $\Rightarrow$ "Frobenius form vanishes"** is clear because of local coordinate form of the submersions.

DEFINITION: The fibers of π are called **leaves**, or **integral submanifolds** of the distribution B. Globally on M, **a leaf of B** is a maximal connected manifold $Z \hookrightarrow M$ which is immersed to M and tangent to B at each point. A distribution for which Frobenius theorem holds is called **integrable**. If B is integrable, the set of its leaves is called **a foliation**. The leaves are manifolds which are immersed to M, but not necessarily closed.

REMARK: To prove the Frobenius theorem for $B \subset TM$, **it suffices to show that each point is contained in an integral submanifold.** In this case, the smooth submersion $U \xrightarrow{\pi} V$ is the projection to the leaf space of B.

The history of Frobenius theorem

History of Frobenius theorem is explained in *"The Mathematics of Frobenius in Context: A Journey Through 18th to 20th Century Mathematics (Sources and Studies in the History of Mathematics and Physical Sciences)", by Thomas Hawkins.*

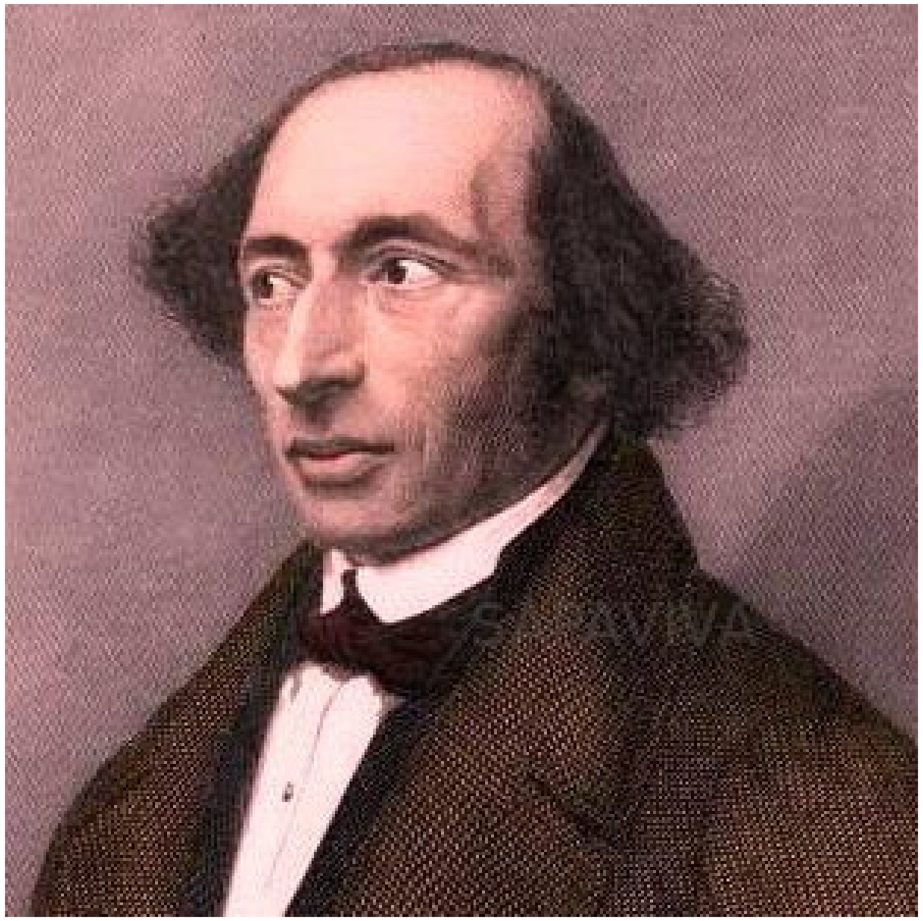
The name "Frobenius theorem" is due to Élie Cartan (1922). Before that it was known as "Pfaff's problem". It was a problem of having "sufficiently many" solutions for a system of differential equations.

In "generic" case was solved by Clebsch (1866), who generalized a weaker result of Jacobi (1837). In 1877, Frobenius gave an equivalent reformulation of Pfaff's problem and solve it, also taking care of the "non-generic" cases omitted by Clebsch.

The technical part of the argument of Frobenius is due to Heinrich Wilhelm Feodor Deahna (1815-1844), who published a version of solution of Pfaff problem in *Crelle's Journal* in 1844.

From the biography of Clebsch (MacTutor History of Mathematics Archive)
...It was just after Clebsch had produced these new ideas that Felix Klein arrived in Göttingen to undertake postdoctoral work with him. The ideas being developed in Clebsh's school had a highly significant influence on Klein. Clebsch's ideas were being rapidly extended by the other talented members of his school. After spending eight months working in Göttingen, Klein spoke to Clebsch about broadening his horizons and spending a semester in Berlin. Clebsch strongly advised him not to go to Berlin and Klein realised that there were serious tensions between the Göttingen school and the one in Berlin. However, he went against Clebsch's advice and spent a semester to Berlin.

Carl Gustav Jacobi (1804-1851), Alfred Clebsch (1833-1872)



Carl Gustav Jacobi,
1804-1851

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Ferdinand Georg Frobenius (1849-1917)

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Frobenius theorem (1)

Proof of the Frobenius theorem. Step 1: Suppose that G is a Lie group acting on a manifold M. Assume that the vector fields from the Lie algebra of G generate a sub-bundle $B \subset TM$. **Then B is integrable, that is, Frobenius theorem holds of $B \subset TM$.** Indeed, the orbits of the G-action are tangent to $B \subset TM$.

Step 2: Let u, v be commuting vector fields on a manifold M, and e^{tu}, e^{tv} be corresponding diffeomorphism flows. **Then e^{tu}, e^{tv} commute.** This easily follows by taking a coordinate system such that u is the coordinate vector field **(do this as an exercise)**.

Step 3: The commutator of vector fields in B belongs to B, however, this does not immediately produce any finite-dimensional Lie algebra: it is not obvious that any subalgebra generated by such vector fields is finite-dimensional. To produce a Lie group with orbits tangent to B, **we need to find a collection $\xi_1, ..., \xi_k \in B$ of vector fields generating B and make sure that the $\xi_1, ..., \xi_k$ generate a finite-dimensional Lie algebra.**

Frobenius theorem (2)

Step 4: The statement of Frobenius Theorem is local, hence we may replace M be a small neighbourhood of a given point. **We are going to show that B locally has a basis of commuting vector fields.** By Step 2, these vector fields can be locally integrated to a commutative group action, and Frobenius Theorem follows from Step 1.

Step 5: Let $\sigma: M \longrightarrow M_1$ be a smooth submersion, $d\sigma: T_xM \longrightarrow T_{\sigma(x)}M_1$ its differential, and $v \in TM$ a vector field which satisfies

$$d\sigma(v)|_x = d\sigma(v)|_y \quad (*)$$

for any $x, y \in \sigma^{-1}(z)$ and any $z \in M_1$. In this case, the vector field $d\sigma(v)$ is well-defined on M_1. **Given two vector fields u and v which satisfy (*), we can easily check that the commutator $[u,v]$ also satisfies (*),** and, moreover, $d\sigma([u,v]) = [d\sigma(u), d\sigma(v)]$.

Frobenius theorem (3)

Step 6: Now we can finish the proof of Frobenius theorem. We need to produce, locally in M, a basis of commuting vector fields $\xi_i \in B$. **We start with producing (locally in M) an auxiliary submersion σ, with the fibers which are complementary to B.** To define such a submersion, we put coordinates locally on M, identifying M with an open subset in $\mathbb{R}^n$, and take a linear map $\sigma: M \longrightarrow M_1 = \mathbb{R}^{\dim B}$ such that $d\sigma: B|_x \longrightarrow T_{\sigma(x)}M_1$ is an isomorphism at some $x \in M$.

Step 7: Then $d\sigma: B|_x \xrightarrow{\sim} T_{\sigma(x)}M_1$ is an isomorphism in a neighbourhood of x; replacing M by a smaller open set, we may assume that $d\sigma: B|_x \xrightarrow{\sim} T_{\sigma(x)}M_1$ is an isomorphism everywhere on M. Let $\zeta_1, ..., \zeta_k$ be the coordinate vector fields on M_1.

Since $d\sigma: B|_x \longrightarrow T_{\sigma(x)}M_1$ is an isomorphism, there exist unique vector fields $\xi_1, ..., \xi_k \in B \subset TM$ such that $d\sigma(\xi_i) = \zeta_i$. By Step 5, $d\sigma([\xi_i, \xi_j]) = [\zeta_i, \zeta_j] = 0$. Since B is involutive, the commutator $[\xi_i, \xi_j]$ is a section of B. Now, the map $d\sigma: B|_x \longrightarrow T_{\sigma(x)}M_1$ is an isomorphism, and therefore the vanishing of $d\sigma([\xi_1, \xi_j])$ implies $[\xi_1, \xi_j] = 0$. **We have constructed a basis of commuting vector fields in B and finished the proof of Frobenius theorem.** ■

Connections

DEFINITION: Recall that **a connection** on a bundle B is an operator $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, where $f \longrightarrow df$ is de Rham differential. When X is a vector field, we denote by $\nabla_X(b) \in B$ the term $\langle \nabla(b), X\rangle$.

REMARK: A connection ∇ on B gives a connection $B^* \xrightarrow{\nabla^*} \Lambda^1 M \otimes B^*$ on the dual bundle, by the formula

$$d(\langle b, \beta\rangle) = \langle \nabla b, \beta\rangle + \langle b, \nabla^* \beta\rangle$$

These connections are usually denoted **by the same letter ∇.**

REMARK: For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on B defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

Curvature

Let $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a vector bundle B. **We extend ∇ to an operator**

$$B \xrightarrow{\nabla} \Lambda^1(M) \otimes B \xrightarrow{\nabla} \Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B \xrightarrow{\nabla} ...$$

using the Leibnitz identity $\nabla(\eta \otimes b) = d\eta \otimes b + (-1)^{\tilde\eta} \eta \wedge \nabla b$.

REMARK: This operation is well defined, because

$$\nabla(\eta \otimes fb) = d\eta \otimes fb + (-1)^{\tilde\eta} \eta \wedge \nabla(fb) =$$
$$d\eta \otimes fb + (-1)^{\tilde\eta} \eta \wedge df \otimes b + f\eta \wedge \nabla b = d(f\eta) \otimes b + f\eta \wedge \nabla b = \nabla(f\eta \otimes b)$$

REMARK: Sometimes $\Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B$ is denoted d_∇.

DEFINITION: The operator $\nabla^2 : B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇.

REMARK: The algebra of differential forms with coefficients in $\operatorname{End} B$ **acts on $\Lambda^* M \otimes B$** via $\eta \otimes a(\eta' \otimes b) = \eta \wedge \eta' \otimes a(b)$, where $a \in \operatorname{End}(B)$, $\eta, \eta' \in \Lambda^* M$, and $b \in B$. **This is the formula expressing the action of ∇^2 on $\Lambda^* M \otimes B$.**

Curvature and commutators

CLAIM: Let $X, Y \in TM$ be vector fields, (B, ∇) a bundle with connection, and $b \in B$ its section. Consider the operator

$$\Theta_B^*(X,Y,b) := \nabla_X \nabla_Y b - \nabla_Y \nabla_X b - \nabla_{[X,Y]} b$$

Then $\Theta_B^*(X,Y,b)$ is linear in all three arguments.

Proof. Step 1: The term $\Theta_B^*(X,Y,fb)$ **has 3 components: one which is C^∞-linear in f, one which takes first derivative and one which takes the second derivative.** The first derivative part is

$\operatorname{Lie}_Y f \nabla_X b + \operatorname{Lie}_X f \nabla_Y b - \operatorname{Lie}_Y f \nabla_X b - \operatorname{Lie}_X f \nabla_Y b - \operatorname{Lie}_{[X,Y]} f b = -\operatorname{Lie}_{[X,Y]} f b,$

the second derivative part is $\operatorname{Lie}_X \operatorname{Lie}_Y(f) b - \operatorname{Lie}_Y \operatorname{Lie}_X(f) b = \operatorname{Lie}_{[X,Y]} f$, they cancel. Therefore, **$\Theta_B^*(X,Y,b)$ is C^∞-linear in b.**

Step 2: Since $[X, fY] = \operatorname{Lie}_X f Y + f[X,Y]$, we have $\nabla_{[X,fY]} b = f \nabla_{[X,Y]} b + \operatorname{Lie}_X f \nabla_Y b$.

Step 3: The term $\Theta_B^*(X, fY, b)$ has two components, f-linear and the component with first derivatives in f. Step 2 implies that the component with derivative of first order is $\operatorname{Lie}_X f \nabla_Y b - \operatorname{Lie}_X f \nabla_Y b = 0$. ■

Curvature and commutators (2)

REMARK:

$$\Theta_B^*(X,Y,b) := \nabla_X\nabla_Y b - \nabla_Y\nabla_X b - \nabla_{[X,Y]}b$$

is another definition of the curvature. **The following theorem shows that it is equivalent to the usual definition**.

THEOREM: Consider $\Theta_B^* : TM \otimes TM \otimes B \longrightarrow B$ as a 2-form with coefficients in $\operatorname{End}(B)$. **Then $\Theta_B^* = \Theta_B$,** where $\Theta_B = \nabla^2$ is the usual curvature.

Proof. Step 1: Since $\Theta_B^*(X,Y)$, $\Theta_B(X,Y)$ are linear in X, Y, it would suffice to prove this equality for coordinate vector fields X, Y.

Step 2: Consider the operator $i_X : \Lambda^i M \otimes B \longrightarrow \Lambda^{i-1} M \otimes B$ of convolution with a vector field X. Writing $\nabla = d + A$, where $A \in \Lambda^1 M \otimes \operatorname{End} B$, we obtain $\nabla_X = \operatorname{Lie}_X + A(X)$, which gives $[\nabla_X, i_Y] = [\operatorname{Lie}_X, i_Y] = 0$ when X, Y are coordinate vector fields.

Step 3:

$$\nabla^2(b)(X,Y) = (i_X i_Y - i_X i_Y)\nabla^2(b) = i_Y\nabla_X\nabla b - i_X\nabla_Y\nabla b = \nabla_X\nabla_Y b - \nabla_Y\nabla_X b.$$

■

Parallel transport along the connection

REMARK: When $M = [0,a]$ is an interval, any bundle B on M is trivial. Let $b_1, ..., b_n$ be a basis in B. Then ∇ can be written as

$$\nabla_{d/dt}\left(\sum f_i b_i\right) = \sum_i \frac{df_i}{dt} b_i + \sum f_i \nabla_{d/dt} b_i$$

with the last term linear on f.

THEOREM: Let B be a vector bundle with connection over $\mathbb{R}$. Then for each $x \in \mathbb{R}$ and each vector $b_x \in B|_x$ **there exists a unique section $b \in B$ such that $\nabla b = 0$, $b|_x = b_x$.**

Proof: This is existence and uniqueness of solutions of an ODE $\frac{db}{dt} + A(b) = 0$.
■

DEFINITION: Let $\gamma: [0,1] \longrightarrow M$ be a smooth path in M connecting x and y, and (B, ∇) a vector bundle with connection. Restricting (B, ∇) to $\gamma([0,1])$, we obtain a bundle with connection on an interval. Solve an equation $\nabla(b) = 0$ for $b \in B\Big|_{\gamma([0,1])}$ and initial condition $b|_x = b_x$. This process is called **parallel transport** along the path via the connection. The vector $b_y := b|_y$ is called **vector obtained by parallel transport of b_x along γ**.

Holonomy group

DEFINITION: (Cartan, 1923) Let (B, ∇) be a vector bundle with connection over M. For each loop γ based in $x \in M$, let $V_{\gamma,\nabla}: B|_x \longrightarrow B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B, ∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

REMARK: Let $B_1 = B^{\otimes n} \otimes (B^*)^{\otimes m}$ be a tensor power of B. The connection on B gives the connection on B_1. Since parallel transport is compatible with the tensor product, **the holonomy representation, associated with B_1, is the corresponding tensor power of $B|_x$.**

DEFINITION: Let B be a vector bundle, and Ψ a section of its tensor power. We say that **connection ∇ preserves Ψ** if $\nabla(\Psi) = 0$. In this case we also say that the tensor Ψ is **parallel** with respect to the connection.

Flat bundles

REMARK: $\nabla(\Psi) = 0$ is equivalent to Ψ being a solution of $\nabla(\Psi) = 0$ on each path γ. This means that **parallel transport preserves Ψ.**

We obtained

COROLLARY: **A section of the tensor power of B is parallel if and only if it is holonomy invariant.**

DEFINITION: A bundle is **flat** if its curvature vanishes.

The following theorem will be proven later today.

THEOREM: Let (B, ∇) be a vector bundle with connection over a simply connected manifold. **Then B is flat if and only if its holonomy group is trivial**.

Fiber of a locally free sheaf

DEFINITION: Recall that **a vector bundle** is a locally free sheaf of modules over $C^\infty M$. A vector bundle is called **trivial** if it is isomorphic to $(C^\infty M)^n$.

DEFINITION: Let $\mathcal{B}$ be an n-dimensional locally free sheaf of C^∞-modules on M, $x \in M$ a point, $\mathfrak{m}_x \subset C^\infty M$ an ideal of $x \in M$ in $C^\infty M$. Define **the fiber** of $\mathcal{B}$ in x as a quotient $\mathcal{B}(M)/\mathfrak{m}\mathcal{B}$. A fiber of $\mathcal{B}$ is denoted $\mathcal{B}|_x$.

REMARK: A fiber of a vector bundle of rank n is an n-dimensional vector space.

REMARK: Let $\mathcal{B} = C^\infty M^n$, and $b \in \mathcal{B}|_x$ a point of a fiber, represented by a germ $\varphi \in \mathcal{B}_x = C^\infty_m M^n$, $\varphi = (f_1, ..., f_n)$. Consider a map Ψ from the set of all fibers $\mathcal{B}$ to $M \times \mathbb{R}^n$, mapping $(x, \varphi = (f_1, ..., f_n))$ to $(f_1(x), ..., f_n(x))$. **Then Ψ is bijective.** Indeed, $\mathcal{B}|_x = \mathbb{R}^n$.

Total space of a vector bundle

DEFINITION: Let $\mathcal{B}$ be an n-dimensional locally free sheaf of C^∞-modules. Denote the set of all vectors in all fibers of $\mathcal{B}$ over all points of M by $\operatorname{Tot}\mathcal{B}$. Let $U \subset M$ be an open subset of M, with $\mathcal{B}|_U$ a trivial bundle. Using the local bijection $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ we consider topology on $\operatorname{Tot}\mathcal{B}$ induced by open subsets in $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ for all open subsets $U \subset M$ and all trivializations of $\mathcal{B}|_U$. Then $\operatorname{Tot}\mathcal{B}$ is called **a total space of a vector bundle $\mathcal{B}$.**

CLAIM: The space $\operatorname{Tot}\mathcal{B}$ with this topology **is a locally trivial fibration over M, with fiber $\mathbb{R}^n$.**

REMARK: Let B be a vector bundle on M, and $\psi \in B^*$ a section of its dual. Then ψ defines a function $x \longrightarrow \langle\psi, x\rangle$ on its total space $\operatorname{Tot}(B) \xrightarrow{\pi} M$, linear on fibers of π. This gives a **bijective correspondence between sections of B^* and functions on $\operatorname{Tot}(B)$ linear on fibers.**

This gives the following claim

CLAIM: Let B be a vector bundle and $\operatorname{Sym}^* B^*$ the direct sum of all symmetric tensor powers of B^*. **Then the ring of sections of $\operatorname{Sym}^* B^*$ is identified with the ring of all smooth functions on $\operatorname{Tot} B \xrightarrow{\pi} M$ which are polynomial on fibers of π.** ■

Polynomial functions on $\operatorname{Tot}(B)$

In Lecture 14, we proved that any derivation of $C^\infty\mathbb{R}^n$ is uniquely determined by its restriction to polynomials:

CLAIM: Let D be the space of derivations $\delta: \mathbb{R}[x_1, ..., x_n] \longrightarrow C^\infty\mathbb{R}^n$. **Then D is the space of derivations of the ring $C^\infty\mathbb{R}^n$.** ■

The same argument brings the following

CLAIM 1: Let D be the space of derivations $\delta: \operatorname{Sym}^* B^* \longrightarrow C^\infty(\operatorname{Tot} B)$. **Then D is the space of derivations of the ring $C^\infty(\operatorname{Tot} B)$.**

Proof: Indeed, any derivation which vanishes on fiberwise polynomial functions vanishes everywhere on $C^\infty(\operatorname{Tot} B)$. ■

Vector fields on $\operatorname{Tot}(B)$

THEOREM: Let (B,∇) be a bundle on M with connection, and $X \in TM$ a vector field. **Then there exists a vector field $\tau_\nabla(X)$ on $\operatorname{Tot}(B)$ mapping a section $u \in \operatorname{Sym}^* B^*$ to $\nabla_X u$.**

Proof: Let $u,v \in \operatorname{Sym}^* B^*$, and $uv \in \operatorname{Sym}^* B^*$ their product. Then $\nabla_x(uv) = u\nabla_x v + v\nabla_x u$ because $\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2)$. Therefore, $\tau_\nabla(X)(u) := \nabla_x(u)$ is a derivation of the ring of functions on $\operatorname{Tot}(B)$ which are polynomial on fibers. By Claim 1, any such derivation can be uniquely extended to a vector field on $\operatorname{Tot}(B)$. ■

DEFINITION: Let (B,∇) be a bundle with connection on M. The corresponding **Ehresmann connection** on $\operatorname{Tot}(B)$ is the distribution $E_\nabla \subset T\operatorname{Tot}(B)$ obtained as $\tau_\nabla(TM)$.

Vector fields on $\operatorname{Tot}(B)$ and parallel sections

CLAIM 2: Let (B, ∇) be a bundle with connection, and $\pi : \operatorname{Tot}(B) \longrightarrow M$ the standard projection, and $T_\pi \operatorname{Tot}(B) = \ker D\pi$ is the vertical tangent space (Lecture 14).

(i) **Then $T \operatorname{Tot} B = E_\nabla \oplus T_\pi \operatorname{Tot}(B)$, where E_∇ is the Ehresmann connection.**

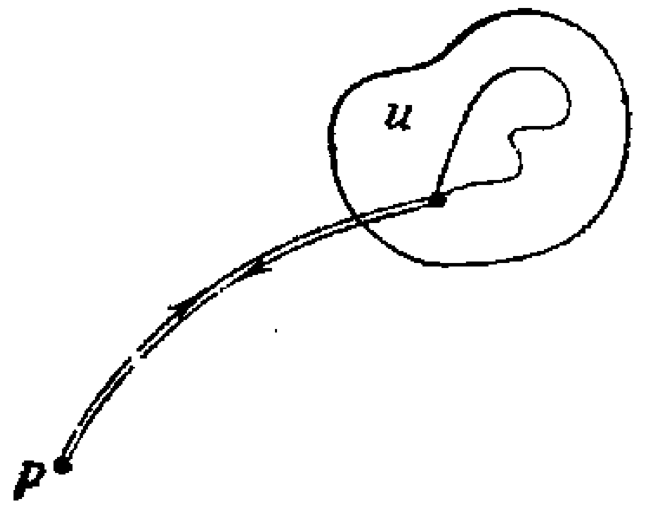
(ii) Moreover, **a section f of B is parallel if an only if its image $f(M) \subset \operatorname{Tot}(B)$ is tangent to E_∇.**

Proof: The second assertion is clear from the definition: **a section b is tangent to E_∇ if it is preserved by all vector fields $a = \tau_\nabla(X)$ generating E_∇.** In this case $\operatorname{Lie}_a(\tilde{b}) = 0$, where $\tilde{b}$ is a function on $\operatorname{Tot}(B^*)$ defined by b. However, $\operatorname{Lie}_a(\tilde{b}) = \widetilde{\nabla_X(b)}$ where $\widetilde{\nabla_X(b)}$ is a function on $\operatorname{Tot}(B^*)$ associated with $\nabla_X(b)$. Therefore, $\operatorname{Lie}_a(\tilde{b}) = 0 \Leftrightarrow \nabla_X(b) = 0$.

To prove (i), we notice that $D\pi\big|_{E_\nabla} : E_\nabla \longrightarrow TM$ is an isomorphism at every point of $\operatorname{Tot} B$. Indeed, these bundles have the same rank, and for each $\tau_\nabla(X) \in E_\nabla$, this vector field acts on functions pulled back from M as Lie_X, hence $D\pi\big|_{E_\nabla}$ is injective. ■

The Lasso lemma

DEFINITION: A **lasso** is a loop of the following form:



The round part is called **a working part** of a loop.

REMARK: ("The Lasso Lemma") Let $\{U_i\}$ be a covering of a manifold, and γ a loop. Then **any contractible loop γ is a product of several lasso, with working part of each inside some U_i.**

Bundles with trivial holonomy

THEOREM: Let (B, ∇) be a vector bundle with connection over a simply connected manifold. **Then B is flat if and only if its holonomy group is trivial**.

Proof: Let B be a flat bundle on M, and $X, Y \in TM$ commuting vector fields. Then $\nabla_X : B \longrightarrow B$ commutes with ∇_Y. Then the Ehresmann connection bundle E_∇ is generated by commuting vector fields $\tau_\nabla(X)$, $\tau_\nabla(Y)$, ..., hence it is involutive. By Frobenius theorem, every point $b \in \operatorname{Tot}(B)$ is contained in a leaf of the corresponding foliation, tangent to E_∇. By Claim 2, such a leaf is a parallel section of B. Therefore, **the holonomy of ∇ around any sufficiently small loop is trivial**. Since $\pi_1(M) = 0$, any contractible loop L can be represented by a composition of lasso with sufficiently small working part. All of them have trivial holonomy, hence L has trivial holonomy as well.

Conversely, assume that B has trivial holonomy. Then $\operatorname{Tot}(B) = M \times B|_x$ because each point is contained in a unique parallel section, hence the bundle E_∇ is involutive. Then $[\nabla_X, \nabla_Y] = 0$ for any commuting $X, Y \in TM$, and the curvature vanishes. ■

Corollary 1: Let B be a flat vector bundle on a simply connected, connected manifold M. **Then for each $x \in M$ and each $b \in B|_x$, there exists a unique parallel section of B passing through b.** ■

Riemann-Hilbert correspondence

THEOREM: The category of locally constant sheaves of vector spaces **is naturally equivalent to the category of vector bundles on M equipped with flat connection.**

Proof. Step 1: Consider a constant sheaf $\mathbb{R}_M$ on M. This is a sheaf of rings, and any locally constant sheaf is a sheaf of $\mathbb{R}_M$-modules.

Let $\mathbb{V}$ be a locally constant sheaf, and $B := \mathbb{V} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$. Since $\mathbb{V}$ is locally constant, the sheaf B is a locally free sheaf of C^∞-modules, that is, a vector bundle. Let $U \subset M$ be an open set such that $\mathbb{V}|_U$ is constant. If $v_1, ..., v_n$ is a basis in $\mathbb{V}(U)$, all sections of $B(U)$ have a form $\sum_{i=1}^n f_i v_i$, where $f_i \in C^\infty U$. Define the connection ∇ by $\nabla\left(\sum_{i=1}^n f_i v_i\right) = \sum df_i \otimes v_i$. This connection is flat because $d^2 = 0$. It is independent from the choice of v_i because v_i is defined canonically up to a matrix with constant coefficients. **We have constructed a functor from locally constant sheaves to flat vector bundles**.

Step 2: Let now (B, ∇) be a flat bundle over M. The functor to locally constant sheaves takes $U \subset M$ and maps it to the space of parallel sections of B over U. This defines a sheaf $\mathbb{B}(U)$. For any simply connected U, and any $x \in M$, the space $\mathbb{B}(U)$ is identified with a vector space $B|_x$ (Corollary 1), hence $\mathbb{B}(U)$ is locally constant. Clearly, $B = \mathbb{B} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$, hence **this construction gives an inverse functor to $\mathbb{V} \mapsto \mathbb{V} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$.** ■