Lecture 19: Riemann-Hilbert correspondence (flat bundles and local systems)

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Connections

DEFINITION: Recall that **a connection** on a bundle B is an operator $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, where $f \longrightarrow df$ is de Rham differential. When X is a vector field, we denote by $\nabla_X(b) \in B$ the term $\langle \nabla(b), X\rangle$.

REMARK: A connection ∇ on B gives a connection $B^* \stackrel{\nabla^*}{\longrightarrow} \Lambda^1 M \otimes B^*$ on the dual bundle, by the formula

$$d(\langle b, \beta\rangle) = \langle \nabla b, \beta\rangle + \langle b, \nabla^* \beta\rangle$$

These connections are usually denoted **by the same letter ∇.**

REMARK: For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on B defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

Curvature

Let $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a vector bundle B. **We extend ∇ to an operator**

$$B \xrightarrow{\nabla} \Lambda^1(M) \otimes B \xrightarrow{\nabla} \Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B \xrightarrow{\nabla} ...$$

using the Leibnitz identity $\nabla(\eta \otimes b) = d\eta \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \nabla b$.

REMARK: This operation is well defined, because

$$\nabla(\eta \otimes fb) = d\eta \otimes fb + (-1)^{\tilde{\eta}} \eta \wedge \nabla(fb) = \\ d\eta \otimes fb + (-1)^{\tilde{\eta}} \eta \wedge df \otimes b + f\eta \wedge \nabla b = d(f\eta) \otimes b + f\eta \wedge \nabla b = \nabla(f\eta \otimes b)$$

REMARK: Sometimes $\Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B$ is denoted d_∇.

DEFINITION: The operator $\nabla^2 : B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇.

REMARK: The algebra of differential forms with coefficients in $\operatorname{End} B$ acts on $\Lambda^* M \otimes B$ via $\eta \otimes a(\eta' \otimes b) = \eta \wedge \eta' \otimes a(b)$, where $a \in \operatorname{End}(B)$, $\eta, \eta' \in \Lambda^* M$, and $b \in B$. **This is the formula expressing the action of ∇^2 on $\Lambda^* M \otimes B$.**

Curvature and commutators

CLAIM: Let $X, Y \in TM$ be vector fields, (B, ∇) a bundle with connection, and $b \in B$ its section. Consider the operator

$$\Theta_B^*(X, Y, b) := \nabla_X \nabla_Y b - \nabla_Y \nabla_X b - \nabla_{[X,Y]} b$$

Then $\Theta_B^*(X, Y, b)$ is linear in all three arguments.

Proof. Step 1: The term $\Theta_B^*(X, Y, fb)$ **has 3 components: one which is C^∞-linear in f, one which takes first derivative and one which takes the second derivative.** The first derivative part is

$\operatorname{Lie}_Y f \nabla_X b + \operatorname{Lie}_X f \nabla_Y b - \operatorname{Lie}_Y f \nabla_X b - \operatorname{Lie}_X f \nabla_Y b - \operatorname{Lie}_{[X,Y]} f b = - \operatorname{Lie}_{[X,Y]} f b,$

the second derivative part is $\operatorname{Lie}_X \operatorname{Lie}_Y (f) b - \operatorname{Lie}_Y \operatorname{Lie}_X (f) b = \operatorname{Lie}_{[X,Y]} f$, they cancel. Therefore, **$\Theta_B^*(X, Y, b)$ is C^∞-linear in b.**

Step 2: Since $[X, fY] = \operatorname{Lie}_X f Y + f[X, Y]$, we have $\nabla_{[X,fY]} b = f \nabla_{[X,Y]} b + \operatorname{Lie}_X f \nabla_Y b$.

Step 3: The term $\Theta_B^*(X, fY, b)$ has two components, f-linear and the component with first derivatives in f. Step 2 implies that the component with derivative of first order is $\operatorname{Lie}_X f \nabla_Y b - \operatorname{Lie}_X f \nabla_Y b = 0$. ■

Curvature and commutators (2)

REMARK:

$$\Theta_B^*(X,Y,b) := \nabla_X\nabla_Y b - \nabla_Y\nabla_X b - \nabla_{[X,Y]}b$$

is another definition of the curvature. **The following theorem shows that it is equivalent to the usual definition**.

THEOREM: Consider $\Theta_B^* : TM \otimes TM \otimes B \longrightarrow B$ as a 2-form with coefficients in $\operatorname{End}(B)$. **Then $\Theta_B^* = \Theta_B$,** where $\Theta_B = \nabla^2$ is the usual curvature.

Proof. Step 1: Since $\Theta_B^*(X,Y)$, $\Theta_B(X,Y)$ are linear in X,Y, it would suffice to prove this equality for coordinate vector fields X,Y.

Step 2: Consider the operator $i_X: \Lambda^i M \otimes B \longrightarrow \Lambda^{i-1} M \otimes B$ of convolution with a vector field X. Writing $\nabla = d + A$, where $A \in \Lambda^1 M \otimes \operatorname{End} B$, we obtain $\nabla_X = \operatorname{Lie}_X + A(X)$, which gives $[\nabla_X, i_Y] = [\operatorname{Lie}_X, i_Y] = 0$ when X,Y are coordinate vector fields.

Step 3:

$$\nabla^2(b)(X,Y) = (i_X i_Y - i_X i_Y)\nabla^2(b) = i_Y\nabla_X\nabla b - i_X\nabla_Y\nabla b = \nabla_X\nabla_Y b - \nabla_Y\nabla_X b.$$

■

Parallel transport along the connection

REMARK: When $M = [0, a]$ is an interval, any bundle B on M is trivial. Let $b_1, ..., b_n$ be a basis in B. Then ∇ can be written as

$$\nabla_{d/dt}\left(\sum f_i b_i\right) = \sum_i \frac{df_i}{dt} b_i + \sum f_i \nabla_{d/dt} b_i$$

with the last term linear on f.

THEOREM: Let B be a vector bundle with connection over $\mathbb{R}$. Then for each $x \in \mathbb{R}$ and each vector $b_x \in B|_x$ **there exists a unique section $b \in B$ such that $\nabla b = 0$, $b|_x = b_x$.**

Proof: This is existence and uniqueness of solutions of an ODE $\frac{db}{dt} + A(b) = 0$.
■

DEFINITION: Let $\gamma : [0, 1] \longrightarrow M$ be a smooth path in M connecting x and y, and (B, ∇) a vector bundle with connection. Restricting (B, ∇) to $\gamma([0, 1])$, we obtain a bundle with connection on an interval. Solve an equation $\nabla(b) = 0$ for $b \in B\Big|_{\gamma([0,1])}$ and initial condition $b|_x = b_x$. This process is called **parallel transport** along the path via the connection. The vector $b_y := b|_y$ is called **vector obtained by parallel transport of b_x along γ**.

Holonomy group

DEFINITION: (Cartan, 1923) Let (B,∇) be a vector bundle with connection over M. For each loop γ based in $x \in M$, let $V_{\gamma,\nabla}: B|_x \longrightarrow B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B,∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

REMARK: Let $B_1 = B^{\otimes n} \otimes (B^*)^{\otimes m}$ be a tensor power of B. The connection on B gives the connection on B_1. Since parallel transport is compatible with the tensor product, **the holonomy representation, associated with B_1, is the corresponding tensor power of $B|_x$.**

DEFINITION: Let B be a vector bundle, and Ψ a section of its tensor power. We say that **connection ∇ preserves Ψ** if $\nabla(\Psi) = 0$. In this case we also say that the tensor Ψ is **parallel** with respect to the connection.

Flat bundles

REMARK: $\nabla(\Psi) = 0$ is equivalent to Ψ being a solution of $\nabla(\Psi) = 0$ on each path γ. This means that **parallel transport preserves Ψ.**

We obtained

COROLLARY: A section of the tensor power of B is parallel if and only if it is holonomy invariant.

DEFINITION: A bundle is **flat** if its curvature vanishes.

The following theorem will be proven later today.

THEOREM: Let (B, ∇) be a vector bundle with connection over a simply connected manifold. **Then B is flat if and only if its holonomy group is trivial**.

Fiber of a locally free sheaf

DEFINITION: Recall that **a vector bundle** is a locally free sheaf of modules over $C^\infty M$. A vector bundle is called **trivial** if it is isomorphic to $(C^\infty M)^n$.

DEFINITION: Let $\mathcal{B}$ be an n-dimensional locally free sheaf of C^∞-modules on M, $x \in M$ a point, $\mathfrak{m}_x \subset C^\infty M$ an ideal of $x \in M$ in $C^\infty M$. Define **the fiber** of $\mathcal{B}$ in x as a quotient $\mathcal{B}(M)/\mathfrak{m}\mathcal{B}$. A fiber of $\mathcal{B}$ is denoted $\mathcal{B}|_x$.

REMARK: A fiber of a vector bundle of rank n is an n-dimensional vector space.

REMARK: Let $\mathcal{B} = C^\infty M^n$, and $b \in \mathcal{B}|_x$ a point of a fiber, represented by a germ $\varphi \in \mathcal{B}_x = C^\infty_m M^n$, $\varphi = (f_1, ..., f_n)$. Consider a map Ψ from the set of all fibers $\mathcal{B}$ to $M \times \mathbb{R}^n$, mapping $(x, \varphi = (f_1, ..., f_n))$ to $(f_1(x), ..., f_n(x))$. **Then Ψ is bijective.** Indeed, $\mathcal{B}|_x = \mathbb{R}^n$.

Total space of a vector bundle

DEFINITION: Let $\mathcal{B}$ be an n-dimensional locally free sheaf of C^∞-modules. Denote the set of all vectors in all fibers of $\mathcal{B}$ over all points of M by $\operatorname{Tot}\mathcal{B}$. Let $U \subset M$ be an open subset of M, with $\mathcal{B}|_U$ a trivial bundle. Using the local bijection $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ we consider topology on $\operatorname{Tot}\mathcal{B}$ induced by open subsets in $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ for all open subsets $U \subset M$ and all trivializations of $\mathcal{B}|_U$. Then $\operatorname{Tot}\mathcal{B}$ is called **a total space of a vector bundle $\mathcal{B}$.**

CLAIM: The space $\operatorname{Tot}\mathcal{B}$ with this topology **is a locally trivial fibration over M, with fiber $\mathbb{R}^n$.**

REMARK: Let B be a vector bundle on M, and $\psi \in B^*$ a section of its dual. Then ψ defines a function $x \longrightarrow \langle \psi, x\rangle$ on its total space $\operatorname{Tot}(B) \xrightarrow{\pi} M$, linear on fibers of π. This gives a **bijective correspondence between sections of B^* and functions on $\operatorname{Tot}(B)$ linear on fibers.**

This gives the following claim

CLAIM: Let B be a vector bundle and $\operatorname{Sym}^* B^*$ the direct sum of all symmetric tensor powers of B^*. **Then the ring of sections of $\operatorname{Sym}^* B^*$ is identified with the ring of all smooth functions on $\operatorname{Tot} B \xrightarrow{\pi} M$ which are polynomial on fibers of π.** ■

Ehresmann connections

DEFINITION: Let $\pi : M \longrightarrow Z$ be a smooth submersion, with $T_\pi M$ **the bundle of vertical tangent vectors** (vectors tangent to the fibers of π). An **Ehresmann connection** on π is a sub-bundle $T_{\text{hor}} M \subset TM$ such that $TM = T_{\text{hor}} M \oplus T_\pi M$. The **parallel transport** along the path $\gamma : [0, a] \longrightarrow Z$ associated with the Ehresmann connection is a diffeomorphism

$$V_t : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(t))$$

smoothly depending on $t \in [0, a]$ and satisfying $\frac{dV_t}{dt} \in T_{\text{hor}} M$.

CLAIM: Let $\pi : M \longrightarrow Z$ be a smooth fibration with compact fibers. Then **the parallel transport, associated with the Ehresmann connection, always exists.**

Proof: Follows from existence and uniqueness of solutions of ODEs. ■

Horizontal lifting

DEFINITION: Let $\pi: M \longrightarrow Z$ be a smooth submersion, and $TM = T_{\text{hor}}M \oplus T_\pi M$ and Ehresmann connection. Given a vector field $X \in TZ$, it **horizontal lifting** is a section $X_{\text{hor}} \in T_{\text{hor}}M$ such that $d\pi$ takes $X_{\text{hor}}|_m$ to $X|_{\pi(m)}$ for all m im M.

CLAIM: For any vector field $X \in TZ$, **its horizontal lifting exists and is unique.**

Proof: Clear. ■

EXERCISE: Prove that **parallel transport along a horizontal lifting mapes fibers of π to fibers.**

COROLLARY: ("Ehresmann's fibration theorem")
Let $\pi: M \longrightarrow M'$ be a smooth submersion of compact manifolds. **Then π is a locally trivial fibration.**

Proof: Fix an Ehresmann connection; it always exists, if we fix a Riemannian metric and write $T_{\text{hor}}M := (T_\pi M)^\perp$. Choose a coordinate system in M', and let $\vec{r} := \sum x_i \frac{d}{dx_i}$ be the radial vector field. Then the parallel transport $e^{-t\vec{r}}$ along $-\vec{r}$ as $t \to \infty$ defines a diffeomorphism between any fiber of π and the fiber over zero; **this trivializes the fibration in a neighbourhood of zero.**
■

Polynomial functions on $\operatorname{Tot}(B)$

In Lecture 14, we proved that any derivation of $\mathbb{C}^\infty\mathbb{R}^n$ is uniquely determined by its restriction to polynomials:

CLAIM: Let D be the space of derivations $\delta: \mathbb{R}[x_1, ..., x_n] \longrightarrow \mathbb{C}^\infty\mathbb{R}^n$. **Then D is the space of derivations of the ring $\mathbb{C}^\infty\mathbb{R}^n$.** ■

The same argument brings the following

CLAIM 1: Let D be the space of derivations $\delta: \operatorname{Sym}^* B^* \longrightarrow \mathbb{C}^\infty(\operatorname{Tot} B)$. **Then D is the space of derivations of the ring $\mathbb{C}^\infty(\operatorname{Tot} B)$.**

Proof: Indeed, any derivation which vanishes on fiberwise polynomial functions vanishes everywhere on $\mathbb{C}^\infty(\operatorname{Tot} B)$. ■

Vector fields on Tot(B)

DEFINITION: Let $\pi: M \longrightarrow Z$ be a smooth submersion, and $X \in TM$ a vector field. Its **lifting** is a vector field $X_1 \in TM$ such that $d\pi(X_1) = X$.

EXERCISE: Prove that X_1 **is a lifting of X if and only if for any function $f \in C^\infty Z$, we have $\pi^*(\operatorname{Lie}_X f) = \operatorname{Lie}_X(\pi^* f)$.**

THEOREM: Let (B, ∇) be a bundle on M with connection, and $X \in TM$ a vector field. **Then there exists a vector field $\tau_\nabla(X)$ on Tot(B) mapping a section $u \in \operatorname{Sym}^* B^*$ to $\nabla_X u$.**

Proof: Let $u, v \in \operatorname{Sym}^* B^*$, and $uv \in \operatorname{Sym}^* B^*$ their product. Then $\nabla_x(uv) = u\nabla_x v + v\nabla_x u$ because $\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2)$. Therefore, $\tau_\nabla(X)(u) := \nabla_x(u)$ is a derivation of the ring of functions on Tot(B) which are polynomial on fibers. By Claim 1, **any such derivation can be uniquely extended to a vector field on Tot(B)**. Moreover, it defines a lifting. ■

DEFINITION: Let (B, ∇) be a bundle with connection on M. The corresponding **Ehresmann connection** on Tot(B) is the distribution $E_\nabla \subset T\operatorname{Tot}(B)$ obtained as $\tau_\nabla(TM)$.

Vector fields on Tot(B) and parallel sections

CLAIM 2: Let (B, ∇) be a bundle with connection, and $\pi: \operatorname{Tot}(B) \longrightarrow M$ the standard projection, and $T_\pi \operatorname{Tot}(B) = \ker D\pi$ is the vertical tangent space (Lecture 14).

(i) **Then $T \operatorname{Tot} B = E_\nabla \oplus T_\pi \operatorname{Tot}(B)$, where E_∇ is constructed as above.**

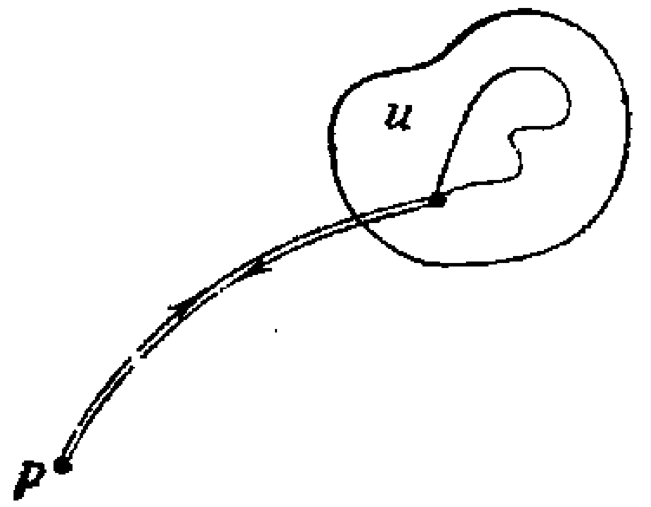
(ii) Moreover, **a section f of B is parallel if an only if its image $f(M) \subset \operatorname{Tot}(B)$ is tangent to E_∇.**

Proof: The second assertion is clear from the definition: **a section b is tangent to E_∇ if it is preserved by all vector fields $a = \tau_\nabla(X)$ generating E_∇.** In this case $\underbrace{\operatorname{Lie}_a(\tilde{b}) = 0}$, where $\underbrace{\tilde{b}}$ is a function on $\operatorname{Tot}(B^*)$ defined by b. However, $\operatorname{Lie}_a(\tilde{b}) = \widetilde{\nabla_X(b)}$ where $\widetilde{\nabla_X(b)}$ is a function on $\operatorname{Tot}(B^*)$ associated with $\nabla_X(b)$. Therefore, $\operatorname{Lie}_a(\tilde{b}) = 0 \Leftrightarrow \nabla_X(b) = 0$.

To prove (i), we notice that $D\pi\big|_{E_\nabla}: E_\nabla \longrightarrow TM$ is an isomorphism at every point of $\operatorname{Tot} B$. Indeed, these bundles have the same rank, and for each $\tau_\nabla(X) \in E_\nabla$, this vector field acts on functions pulled back from M as Lie_X, hence $D\pi\big|_{E_\nabla}$ is injective. ■

The Lasso lemma

DEFINITION: A **lasso** is a loop of the following form:



The round part is called **a working part** of a loop.

REMARK: ("The Lasso Lemma") Let $\{U_i\}$ be a covering of a manifold, and γ a loop. Then **any contractible loop γ is a product of several lasso, with working part of each inside some U_i.**

Bundles with trivial holonomy

THEOREM: Let (B, ∇) be a vector bundle with connection over a simply connected manifold. **Then B is flat if and only if its holonomy group is trivial**.

Proof: Let B be a flat bundle on M, and $X, Y \in TM$ commuting vector fields. Then $\nabla_X : B \longrightarrow B$ commutes with ∇_Y. Then the Ehresmann connection bundle E_∇ is generated by commuting vector fields $\tau_\nabla(X)$, $\tau_\nabla(Y)$, ..., hence it is involutive. By Frobenius theorem, every point $b \in \operatorname{Tot}(B)$ is contained in a leaf of the corresponding foliation, tangent to E_∇. By Claim 2, such a leaf is a parallel section of B. Therefore, **the holonomy of ∇ around any sufficiently small loop is trivial**. Since $\pi_1(M) = 0$, any contractible loop L can be represented by a composition of lasso with sufficiently small working part. All of them have trivial holonomy, hence L has trivial holonomy as well.

Conversely, assume that B has trivial holonomy. Then $\operatorname{Tot}(B) = M \times B|_x$ because each point is contained in a unique parallel section, hence the bundle E_∇ is involutive. Then $[\nabla_X, \nabla_Y] = 0$ for any commuting $X, Y \in TM$, and the curvature vanishes. ■

Corollary 1: Let B be a flat vector bundle on a simply connected, connected manifold M. **Then for each $x \in M$ and each $b \in B|_x$, there exists a unique parallel section of B passing through b.** ■

Riemann-Hilbert correspondence

THEOREM: The category of locally constant sheaves of vector spaces **is naturally equivalent to the category of vector bundles on M equipped with flat connection.**

Proof. Step 1: Consider a constant sheaf $\mathbb{R}_M$ on M. This is a sheaf of rings, and any locally constant sheaf is a sheaf of $\mathbb{R}_M$-modules.

Let $\mathbb{V}$ be a locally constant sheaf, and $B := \mathbb{V} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$. Since $\mathbb{V}$ is locally constant, the sheaf B is a locally free sheaf of C^∞-modules, that is, a vector bundle. Let $U \subset M$ be an open set such that $\mathbb{V}|_U$ is constant. If $v_1, ..., v_n$ is a basis in $\mathbb{V}(U)$, all sections of $B(U)$ have a form $\sum_{i=1}^n f_i v_i$, where $f_i \in C^\infty U$. Define the connection ∇ by $\nabla\left(\sum_{i=1}^n f_i v_i\right) = \sum df_i \otimes v_i$. This connection is flat because $d^2 = 0$. It is independent from the choice of v_i because v_i is defined canonically up to a matrix with constant coefficients. **We have constructed a functor from locally constant sheaves to flat vector bundles**.

Step 2: Let now (B, ∇) be a flat bundle over M. The functor to locally constant sheaves takes $U \subset M$ and maps it to the space of parallel sections of B over U. This defines a sheaf $\mathbb{B}(U)$. For any simply connected U, and any $x \in M$, the space $\mathbb{B}(U)$ is identified with a vector space $B|_x$ (Corollary 1), hence $\mathbb{B}(U)$ is locally constant. Clearly, $B = \mathbb{B} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$, hence **this construction gives an inverse functor to $\mathbb{V} \mapsto \mathbb{V} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$.** ■