Lecture 20: Riemann-Hilbert correspondence (flat bundles and local systems)

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Ehresmann connections (reminder)

DEFINITION: Let $\pi : M \longrightarrow Z$ be a smooth submersion, with $T_\pi M$ **the bundle of vertical tangent vectors** (vectors tangent to the fibers of π). An **Ehresmann connection** on π is a sub-bundle $T_{\mathsf{hor}} M \subset TM$ such that $TM = T_{\mathsf{hor}} M \oplus T_\pi M$. The **parallel transport** along the path $\gamma : [0, a] \longrightarrow Z$ associated with the Ehresmann connection is a diffeomorphism

$$V_t : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(t))$$

smoothly depending on $t \in [0, a]$ and satisfying $\frac{dV_t}{dt} \in T_{\mathsf{hor}} M$.

CLAIM: Let $\pi : M \longrightarrow Z$ be a smooth fibration with compact fibers. Then **the parallel transport, associated with the Ehresmann connection, always exists.**

Proof: Follows from existence and uniqueness of solutions of ODEs. ∎

Horizontal lifting (reminder)

DEFINITION: Let $\pi: M \longrightarrow Z$ be a smooth submersion, and $TM = T_{hor}M \oplus T_\pi M$ and Ehresmann connection. Given a vector field $X \in TZ$, it **horizontal lifting** is a section $X_{hor} \in T_{hor}M$ such that $d\pi$ takes $X_{hor}|_m$ to $X|_{\pi(m)}$ for all m im M.

CLAIM: For any vector field $X \in TZ$, **its horizontal lifting exists and is unique.**

Proof: Clear. ■

EXERCISE: Prove that **parallel transport along a horizontal lifting mapes fibers of π to fibers.**

COROLLARY: ("Ehresmann's fibration theorem")
Let $\pi: M \longrightarrow M'$ be a smooth submersion of compact manifolds. **Then π is a locally trivial fibration.**

Proof: Fix an Ehresmann connection; it always exists, if we fix a Riemannian metric and write $T_{hor}M := (T_\pi M)^\perp$. Choose a coordinate system in M', and let $\vec{r} := \sum x_i \frac{d}{dx_i}$ be the radial vector field. Then the parallel transport $e^{-t\vec{r}}$ along $-\vec{r}$ as $t \to \infty$ defines a diffeomorphism between any fiber of π and the fiber over zero; **this trivializes the fibration in a neighbourhood of zero.**
■

Polynomial functions on $\operatorname{Tot}(B)$

In Lecture 14, we proved that any derivation of $\mathbb{C}^\infty\mathbb{R}^n$ is uniquely determined by its restriction to polynomials:

CLAIM: Let D be the space of derivations $\delta: \mathbb{R}[x_1, ..., x_n] \longrightarrow \mathbb{C}^\infty\mathbb{R}^n$. **Then D is the space of derivations of the ring $\mathbb{C}^\infty\mathbb{R}^n$.** ■

The same argument brings the following

CLAIM 1: Let D be the space of derivations $\delta: \operatorname{Sym}^* B^* \longrightarrow \mathbb{C}^\infty(\operatorname{Tot} B)$. **Then D is the space of derivations of the ring $\mathbb{C}^\infty(\operatorname{Tot} B)$.**

Proof: Indeed, any derivation which vanishes on fiberwise polynomial functions vanishes everywhere on $\mathbb{C}^\infty(\operatorname{Tot} B)$. ■

Vector fields on Tot(B)

DEFINITION: Let $\pi : M \longrightarrow Z$ be a smooth submersion, and $X \in TM$ a vector field. Its **lifting** is a vector field $X_1 \in TM$ such that $d\pi(X_1) = X$.

EXERCISE: Prove that X_1 **is a lifting of** X **if and only if for any function** $f \in C^\infty Z$**, we have** $\pi^*(\operatorname{Lie}_X f) = \operatorname{Lie}_X(\pi^* f)$**.**

THEOREM: Let (B, ∇) be a bundle on M with connection, and $X \in TM$ a vector field. **Then there exists a vector field** $\tau_\nabla(X)$ **on** Tot(B) **mapping a section** $u \in \operatorname{Sym}^* B^*$ **to** $\nabla_X u$**.**

Proof: Let $u, v \in \operatorname{Sym}^* B^*$, and $uv \in \operatorname{Sym}^* B^*$ their product. Then $\nabla_x(uv) = u\nabla_x v + v\nabla_x u$ because $\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2)$. Therefore, $\tau_\nabla(X)(u) := \nabla_x(u)$ is a derivation of the ring of functions on Tot(B) which are polynomial on fibers. By Claim 1, **any such derivation can be uniquely extended to a vector field on** Tot(B). Moreover, it defines a lifting. ■

DEFINITION: Let (B, ∇) be a bundle with connection on M. The corresponding **Ehresmann connection** on Tot(B) is the distribution $E_\nabla \subset T\,\mathrm{Tot}(B)$ obtained as $\tau_\nabla(TM)$.

Vector fields on $\operatorname{Tot}(B)$ and parallel sections

CLAIM 2: Let (B, ∇) be a bundle with connection, and $\pi: \operatorname{Tot}(B) \longrightarrow M$ the standard projection, and $T_\pi \operatorname{Tot}(B) = \ker D\pi$ is the vertical tangent space (Lecture 14).

(i) **Then $T \operatorname{Tot} B = E_\nabla \oplus T_\pi \operatorname{Tot}(B)$, where E_∇ is constructed as above.**

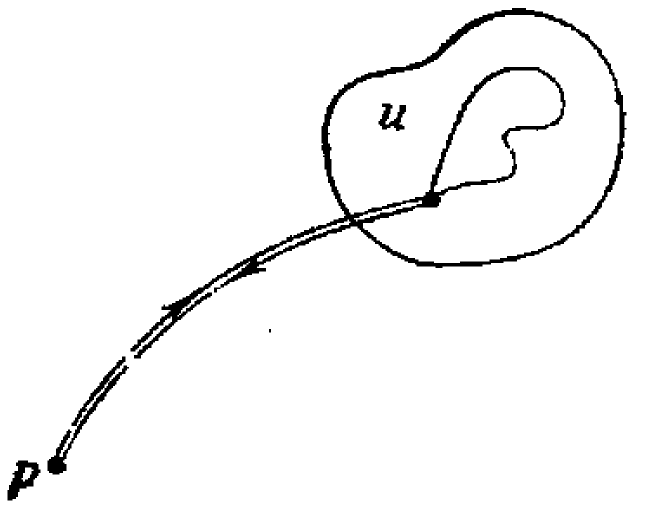
(ii) Moreover, **a section f of B is parallel if an only if its image $f(M) \subset \operatorname{Tot}(B)$ is tangent to E_∇.**

Proof: The second assertion is clear from the definition: **a section b is tangent to E_∇ if it is preserved by all vector fields $a = \tau_\nabla(X)$ generating E_∇.** In this case $\operatorname{Lie}_a(\tilde{b}) = 0$, where $\tilde{b}$ is a function on $\operatorname{Tot}(B^*)$ defined by b. However, $\operatorname{Lie}_a(\tilde{b}) = \widetilde{\nabla_X(b)}$ where $\widetilde{\nabla_X(b)}$ is a function on $\operatorname{Tot}(B^*)$ associated with $\nabla_X(b)$. Therefore, $\operatorname{Lie}_a(\tilde{b}) = 0 \Leftrightarrow \nabla_X(b) = 0$.

To prove (i), we notice that $D\pi\big|_{E_\nabla}: E_\nabla \longrightarrow TM$ is an isomorphism at every point of $\operatorname{Tot} B$. Indeed, these bundles have the same rank, and for each $\tau_\nabla(X) \in E_\nabla$, this vector field acts on functions pulled back from M as Lie_X, hence $D\pi\big|_{E_\nabla}$ is injective. ■

The Lasso lemma

DEFINITION: A **lasso** is a loop of the following form:



The round part is called **a working part** of a loop.

REMARK: ("The Lasso Lemma") Let $\{U_i\}$ be a covering of a manifold, and γ a loop. Then **any contractible loop γ is a product of several lasso, with working part of each inside some U_i.**

Bundles with trivial holonomy

THEOREM: Let (B, ∇) be a vector bundle with connection over a simply connected manifold. **Then B is flat if and only if its holonomy group is trivial**.

Proof: Let B be a flat bundle on M, and $X, Y \in TM$ commuting vector fields. Then $\nabla_X : B \longrightarrow B$ commutes with ∇_Y. Then the Ehresmann connection bundle E_∇ is generated by commuting vector fields $\tau_\nabla(X)$, $\tau_\nabla(Y)$, ..., hence it is involutive. By Frobenius theorem, every point $b \in \operatorname{Tot}(B)$ is contained in a leaf of the corresponding foliation, tangent to E_∇. By Claim 2, such a leaf is a parallel section of B. Therefore, **the holonomy of ∇ around any sufficiently small loop is trivial**. Since $\pi_1(M) = 0$, any contractible loop L can be represented by a composition of lasso with sufficiently small working part. All of them have trivial holonomy, hence L has trivial holonomy as well.

Conversely, assume that B has trivial holonomy. Then $\operatorname{Tot}(B) = M \times B|_x$ because each point is contained in a unique parallel section, hence the bundle E_∇ is involutive. Then $[\nabla_X, \nabla_Y] = 0$ for any commuting $X, Y \in TM$, and the curvature vanishes. ■

Corollary 1: Let B be a flat vector bundle on a simply connected, connected manifold M. **Then for each $x \in M$ and each $b \in B|_x$, there exists a unique parallel section of B passing through b.** ■

Riemann-Hilbert correspondence

THEOREM: The category of locally constant sheaves of vector spaces **is naturally equivalent to the category of vector bundles on M equipped with flat connection.**

Proof. Step 1: Consider a constant sheaf $\mathbb{R}_M$ on M. This is a sheaf of rings, and any locally constant sheaf is a sheaf of $\mathbb{R}_M$-modules.

Let $\mathbb{V}$ be a locally constant sheaf, and $B := \mathbb{V} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$. Since $\mathbb{V}$ is locally constant, the sheaf B is a locally free sheaf of C^∞-modules, that is, a vector bundle. Let $U \subset M$ be an open set such that $\mathbb{V}|_U$ is constant. If $v_1, ..., v_n$ is a basis in $\mathbb{V}(U)$, all sections of $B(U)$ have a form $\sum_{i=1}^n f_i v_i$, where $f_i \in C^\infty U$. Define the connection ∇ by $\nabla\left(\sum_{i=1}^n f_i v_i\right) = \sum df_i \otimes v_i$. This connection is flat because $d^2 = 0$. It is independent from the choice of v_i because v_i is defined canonically up to a matrix with constant coefficients. **We have constructed a functor from locally constant sheaves to flat vector bundles**.

Step 2: Let now (B, ∇) be a flat bundle over M. The functor to locally constant sheaves takes $U \subset M$ and maps it to the space of parallel sections of B over U. This defines a sheaf $\mathbb{B}(U)$. For any simply connected U, and any $x \in M$, the space $\mathbb{B}(U)$ is identified with a vector space $B|_x$ (Corollary 1), hence $\mathbb{B}(U)$ is locally constant. Clearly, $B = \mathbb{B} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$, hence **this construction gives an inverse functor to $\mathbb{V} \mapsto \mathbb{V} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$.** ■

Torsion

REMARK: **"Connection on a manifold M"** denotes a connection on the bundle TM or $\Lambda^1 M$. Such a connection **induces a connection on all its tensor powers $TM^{\otimes i} \otimes \Lambda^1 M^{\otimes j}$** by Leibnitz rule.

DEFINITION: Let ∇ be a connection on $\Lambda^1 M$,

$$\Lambda^1 \xrightarrow{\nabla} \Lambda^1 M \otimes \Lambda^1 M.$$

The torsion of ∇ is a map $T_\nabla : \Lambda^1 M \longrightarrow \Lambda^2 M$ defined as $\nabla \circ \operatorname{Alt} - d$, where $\operatorname{Alt} : \Lambda^1 M \otimes \Lambda^1 M \longrightarrow \Lambda^2 M$ is exterior multiplication.

REMARK:

$$\begin{aligned} T_\nabla(f\eta) &= \operatorname{Alt}(f\nabla\eta + df \otimes \eta) - d(f\eta) \\ &= f\Big[\operatorname{Alt}(\nabla\eta) - d\eta\Big] + df \wedge \eta - df \wedge \eta = fT_\nabla(\eta). \end{aligned}$$

Therefore T_∇ is linear.

Torsion and commutator of vector fields

REMARK: Cartan formula gives

$$\begin{aligned} T_\nabla(\eta)(X,Y) =& \nabla_X(\eta)(Y) - \nabla_Y(\eta)(X) - d\eta(X,Y) \\ =& \nabla_X(\eta)(Y) - \nabla_Y(\eta)(X) - \eta([X,Y]) - \operatorname{Lie}_X(\eta(Y)) + \operatorname{Lie}_Y(\eta(X)). \end{aligned}$$

On the other hand, $\nabla_X(\eta)(Y) = \operatorname{Lie}_X(\eta(Y)) - \eta(\nabla_X(Y))$. Comparing the equations, we obtain

$$T_\nabla(\eta)(X,Y) = \eta\Big(\nabla_X(Y) - \nabla_Y(X) - [X,Y]\Big).$$

Torsion is often defined as a map $\Lambda^2 TM \longrightarrow TM$ using the formula $\nabla_X(Y) - \nabla_Y(X) - [X,Y]$.

We have just proved

CLAIM: The tensor $\nabla_X(Y) - \nabla_Y(X) - [X,Y]$ is dual to the torsion map $\nabla \circ \operatorname{Alt} - d: \ \Lambda^1 M \longrightarrow \Lambda^2 M$ defined above.

Flat affine manifolds

DEFINITION: Affine map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a composition of a linear map and a parallel translation.

DEFINITION: A **flat affine manifold** is a manifold M equipped with an atlas $\{U_i\}$ such that all transition maps are affine. In this case, U_i are called **affine charts**.

REMARK: Let M be a flat affine manifold, U an affine chart. Consider the basis in $\Lambda^1 U$ given by the coordinate 1-forms $dx_1, ..., dx_n$. Any affine map puts dx_i to a linear combination of coordinate 1-forms, hence the subsheaf in $\Lambda^1 M$ sheaf generated by dx_i is locally constant. **Riemann-Hilbert correspondence gives a natural flat connection $\nabla: \Lambda^1 M \longrightarrow \Lambda^1 M \otimes \Lambda^1 M$ such that $\nabla(dx_i) = 0$.**

THEOREM: Let M be a flat affine manifold, and ∇ a flat connection on M constructed above. **Then ∇ is torsion-free.** Moreover, **every torsion-free flat connection is obtained from a flat affine structure this way.**

Flat affine manifolds and torsion-free connections

THEOREM: Let M be a flat affine manifold, and ∇ a flat connection on M constructed above. **Then ∇ is torsion-free.** Moreover, **every torsion-free flat connection is obtained from a flat affine structure this way.**

Proof. Step 1: Consider a bundle B over M trivialized by a frame $b_1, ..., b_n$. Then **there exists a unique connection ∇ such that $\nabla(b_i) = 0$.** Indeed, $\nabla\left(\sum_{i=1}^n f_i b_i\right) = \sum df_i \otimes b_i$.

Step 2: An affine structure gives a torsion-free flat connection as follows. Let $x_1, ..., x_n$ be flat affine coordinates on $U \subset M$. Then $dx_1, ..., dx_n$ is a frame trivializing $\Lambda^1 U$, and we can define a connection ∇ such that $\nabla(dx_i) = 0$ as in Step 1. Any affine transform maps the form dx_i to a linear combination of dx_i. Therefore, for any other set of coordinates $y_1, ..., y_n$ defining the same affine structure, one has $\nabla(dy_i) = 0$. This implies that **∇ is independent on the choice of coordinates.** It is flat because $\nabla^2(dx_i) = 0$ and torsion-free because $\operatorname{Alt}(\nabla\left(\sum_{i=1}^n f_i dx_i\right) = \sum df_i \wedge dx_i = d\left(\sum_{i=1}^n f_i dx_i\right)$.

Flat affine manifolds and torsion-free connections (2)

THEOREM: Let M be a flat affine manifold, and ∇ a flat connection on TM constructed above. **Then ∇ is torsion-free.** Moreover, **every torsion-free flat connection is obtained from a flat affine structure this way.**

Step 3: It remains to show that every torsion-free, flat connection ∇ on M is obtained this way. By Riemann-Hilbert correspondence (Lecture 17) in a neighbourhood of each point there exists a frame $b_1, ..., b_n \in \Lambda^1 M$ such that $\nabla(b_i) = 0$. Since $\operatorname{Alt}(\nabla b_i) = db_i = 0$, each form b_i is closed. Poincaré lemma implies that $b_i = dx_i$. Since the forms dx_i are linearly independent, the derivative of the map $\kappa(m) := (x_1(m), ..., x_n(m))$ is invertible. Then κ is locally a diffeomorphism to $\mathbb{R}^n$, and x_i are coordinates. Clearly, ∇ is a connection constructed from these coordinates as in Step 2. We obtained an atlas on M such that $\nabla(dx_i) = 0$ for each coordinate function x_i. **It remains only to show that this atlas defines a flat affine structure**.

Step 4: Clearly, $\nabla\left(\sum_{i=1}^n f_i dx_i\right) = 0$ if and only if all f_i are constant. The transition functions between coordinates map $m = (x_1, ..., x_n)$ to $y_i = \sum_{i=1}^n \varphi_i(m)$ such that $\nabla(dy_i) = 0$. Expressing each dy_i as $dy_i = \sum_{i=1}^n f_{ij} dx_i$ and using $0 = \nabla\left(\sum_{i=1}^n f_{ij} dx_i\right) = df_{ik} \otimes dx_i$, we obtain that all functions f_{ij} (partial derivatives of the transition functions $\varphi_i(m)$) are constant. A function with constant partial derivatives is always affine, hence **the transition functions between charts are affine.** ■