Seiberg-Witten Invariants 1: Clifford algebras

Rules: It is recommended to solve all unmarked and !-problems or to find the solution online. It's better to do it in order starting from the beginning, because the solutions are often contained in previous problems. The problems with * are harder, and ** are very hard; don't be disappointed if you can't solve them, but feel free to try. Have fun! To complete the exercises, write them up and bring it to the class, and then we can schedule an appointment to discuss the solututions.

1.1 Clifford algebras

Remark 1.1. Throughout this assignment, (V, q) is a finite-dimensional vector space over $\mathbb{R}$ equipped with a scalar product.

Definition 1.1. **Clifford algebra** $\operatorname{Cl}(V)$ is the quotient of the tensor algebra $\bigoplus_{i=0}^\infty T^{\otimes i}V$ by a two-sided ideal $\mathcal{J}$ generated by the relation $xy + yx + q(x,y)1$, for all $x, y \in V$.

Exercise 1.1. (universal property of Clifford algebras)
Let A be an associative algebra with unity, and $u: V \longrightarrow A$ a linear map. Assume that $u(v)^2 = -q(v,v)1$. Prove that u is extended to an algebra homomorphism $\operatorname{Cl}(V) \longrightarrow A$, and this extension is unique. Prove that any linear map $h \in O(V)$ can be extended to an automorphism of the Clifford algebra.

Exercise 1.2. Let $\operatorname{Cl}^*(V)$ be the multiplicative group of the Clifford algebra. Define **the adjoint action** $\operatorname{Ad}: \operatorname{Cl}^*(V) \longrightarrow \operatorname{Aut}(\operatorname{Cl}(V))$ which takes $\phi \in \operatorname{Cl}^*(V)$ to the automorphism $x \mapsto \phi x \phi^{-1}$. Let $\mathsf{ad}: \operatorname{Cl}(V) \longrightarrow \mathfrak{gl}(\operatorname{Cl}(V))$ be the Lie algebra representation taking $z \in \operatorname{Cl}(V))$ to $x \mapsto [x, z]$. Prove that $e^{\mathsf{ad}_x} = \operatorname{Ad}_{e^x}$, where $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ is the exponent map in the Clifford algebra.

Exercise 1.3. Let $v \in V$ be a vector which satisfies $q(v,v) \neq 0$.

a. Prove that $v^{-1} = -q(v,v)^{-1}v$.

b. Prove that $vuv = q(v,v)u - 2q(v,u)v$

c. Prove that $\operatorname{Ad}_v(u) = -u + 2\frac{q(v,u)}{q(v,v)}v$.

d. Prove that $r_v: u \mapsto -u + 2\frac{q(v,u)}{q(v,v)}v$ belongs to $O(V)$ and satisfies $r_v^2 = \mathsf{Id}$.

e. Prove that the group $O(V)$ acts on $\operatorname{Cl}(V)$ naturally by algebra automorphisms.

f. Prove that the action of Ad_v on $\operatorname{Cl}(V)$ is induced by the reflection $u \mapsto -u + 2\frac{q(v,u)}{q(v,v)}v$.

Exercise 1.4. Let $u, v \in V$ be non-collinear vectors which satisfy $q(v,v), q(u,u) \neq 0$ and generate a 2-plane with non-degenerate scalar product. Prove that $\operatorname{Ad}_v \operatorname{Ad}_u$ acts on $\operatorname{Cl}(V)$ as a rotation in the plane $\langle u, v\rangle$.

Hint. Use the previous exercise.

Exercise 1.5. Let $u, v \in V$ be orthogonal vectors which satisfy $q(v,v), q(u,u) \neq 0$. Prove that ad_{uv-vu}, for all u, v, generate the action of $\mathfrak{so}(V)$ on $\mathfrak{gl}(\operatorname{Cl}(V))$.

Hint. Use the previous exercise.

Exercise 1.6. Consider the isometry $v \mapsto -v$ in V, extend it to an automorphism $\iota \in \operatorname{Aut}(\operatorname{Cl}(V))$, and let $\operatorname{Cl}(V)_{ev} := \{\phi \in \operatorname{Cl}(V) \mid \iota(\phi) = \phi\}$. Prove that $\dim \operatorname{Cl}(V)_{ev} = \frac{1}{2} \dim \operatorname{Cl}(V)$.

1.2 Spin group

Definition 1.2. Let $\operatorname{Pin}(V, q)$ be the subgroup of $\operatorname{Cl}^*(V)$ generated by Ad_v, for all $v \in V$, $q(v, v) \neq 0$. Let $\operatorname{Spin}(V, q) := \operatorname{Pin}(V, q) \cap \operatorname{Cl}(V)_{ev}$. This group is called **the spin group**.

Exercise 1.7. Let q be a non-degenerate scalar product on V.

a. Prove that $\operatorname{Pin}(V, q)$ is not connected.

b. (*) Prove that $\operatorname{Spin}(V, q)$ is connected when q is sign-definite.

Exercise 1.8 (*). Suppose that $p, q > 0$. Prove that $O(p, q)$ has 4 connected components.

Exercise 1.9 (*). Suppose that $p, q > 1$. Let $SO^+(p, q)$ be the connected component of $SO(p, q)$. Prove that $SO^+(p, q)$ is homotopy equivalent to $SO(p) \times SO(q)$.

Exercise 1.10. Let $Q \subset \mathbb{R}^{p+q}$ be the hyperboloid $Q := \{(x_1, ..., x_{p+q}) \mid \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2 = 1\}$.

a. Prove that $SO(p, q)$ acts transitively on Q, and the stabilizer of a point is $SO(p-1, q)$.

b. (!) Prove that Q is homotopy equivalent to the sphere S^{p-1}.

c. (!) Assume that $p > 1, q > 0$. Prove that $Q = \frac{SO^+(p,q)}{SO^+(p-1,q)}$.

Hint. Construct a deformation retraction of Q to the set $\sum_{i=1}^{p} x_i^2 = 1$.

Exercise 1.11. Let $p > 2$. Prove that $\pi_1(SO^+(p, q)) = \pi_1(SO^+(p-1, q))$.

Hint. Use the previous exercise and the homotopy exact sequence associated with the fibration $SO^+(p, q) \mapsto Q$ with the fiber $SO^+(p-1, q)$.

Exercise 1.12 (!). Let $p > 2$. Prove that $\pi_1(SO^+(p, q)) = \mathbb{Z}/2$.

Hint. Use the previous exercise.

Remark 1.2. We denote the Clifford algebra for the form q of signature (m, n) by $\operatorname{Cl}(m, n)$.

Exercise 1.13. Prove that $\operatorname{Cl}(0, 2) \cong \operatorname{Cl}(1, 1) \cong \operatorname{Mat}(2, \mathbb{R})$ and $\operatorname{Cl}(2, 0) \cong \mathbb{H}$.

Exercise 1.14 (*). Prove that the image of $\mathfrak{so}(0, 3) = \operatorname{Lie} \operatorname{Spin}(0, 3)$ in $\operatorname{Cl}(0, 3) = \operatorname{Mat}(2, \mathbb{C})$ is generated by the quaternionic rotation matrices. Use this observation to show that $\operatorname{Spin}(0, 3) \cong SU(2)$.

Exercise 1.15 (*). Prove that $\operatorname{Spin}(0, n)$ is a non-trivial double cover of $SO(n)$ for all $n \geqslant 3$.