# Seiberg-Witten Invariants 2: Principal bundles

## 2.1 Quotient topology

**Definition 2.1.** Let $M$ be a topological space, and $\sim$ an equivalence relation. The set of equivalence classes is denoted $M/\sim$. We define **the quotient topology** on $M/\sim$ as follows: a subset $U \subset M/\sim$ is open if and only if its preimage in $M$ is open. A group action on $M$ defines the following equivalence relation: $x \sim y$ if $x, y$ belong to the same orbit of $G$. The set $M/G$ of $G$-orbits with the quotient topology is called **the quotient space**

**Definition 2.2.** Let $G$ be a topological group which acts on a topological space $M$. We say that this action is **continuous**, when the corresponding map $G \times M \to M$ is continuous.

**Exercise 2.1.** Prove that any action of a discrete group is continuous.

**Exercise 2.2.** Let $G$ be a finite group acting on a Hausdorff space $M$. Prove that $M/G$ is Hausdorff.

**Hint.** Let $x, y \in M$ two points on different orbits of $G/$ Take non-intersecting neighbourhoods $U \ni x, U' \ni y$, and let $W := \bigcap_{g\in G} gU$, $W' := \bigcap_{g\in G} gU'$.

**Exercise 2.3 (*).** Let $G$ be a compact group acting on a Hausdorff space $M$. Prove that $M/G$ is Hausdorff.

**Exercise 2.4.** Let $G = \mathbb{Z}$ freely act on a compact manifold $M$ of positive dimension. Prove that $M/G$ is never Hausdorff.

## 2.2 Principal bundles

**Definition 2.3.** Let $M$ be a topological space, and $G$ a topological group which acts on $M$ freely and continuously. Assume that $M/G$ with the quotient topology is Hausdorff. Then the fibration $\pi: \ M \to M/G$ equipped with the natural $G$-cation is called **a principal bundle**. If, in addition, $M$ and $M/G$ are equipped with smooth structures, $G \times M \to M$ is smooth, and $\pi$ is a submersion (that is, its differential is surjective in each point), we say that $\pi: \ M \to M/G$ is **a smooth principal bundle.** In this handout, all principal bundles are assumed to be smooth.

**Exercise 2.5 (!).** Let $G$ be a compact Lie group acting freely on a smooth manifold $M$. Prove that the quotient space $M/G$ is a smooth manifold, and the map $\pi: \ M \to M/G$ is a smooth principal bundle.

**Definition 2.4. The Hopf fibration** is the restriction of the standard projection $\mathbb{C}^2\backslash 0 \to \mathbb{C}P^1$ to the unit sphere $S^3 \subset \mathbb{C}^2$.

**Exercise 2.6.** Prove that the Hopf fibration is a principal $S^1$-bundle.

**Exercise 2.7.** Consider a smooth map $S^{2n} \to M$ to a manifold of smaller dimension. Can it be a principal bundle?

**Exercise 2.8.** Construct a principal $SU(2)$-bundle $S^7 \to S^4$.

**Exercise 2.9 (*).** Consider a free action of $\mathbb{R}$ on a compact manifold $M$. Prove that the quotient $M/\mathbb{R}$ is never Hausdorff.