

SW Invariants 3: Serre's spectral sequence

I am going to describe the spectral sequence as a computational recipe, with no background given. The reader needs to learn how to apply the definition of spectral sequence (Definition 2.1) to compute the cohomology. In the sequel $\pi : E \rightarrow B$ is a locally trivial fibration (or, more generally, a Serre's fibration), with fiber F . We assume that the monodromy action of $\pi_1(B)$ on $H^*(F)$ is trivial. We also assume that the cohomology are taken with coefficients in a field of characteristic 0. However, the key statements can be easily refashioned to obtain results about cohomology with coefficients in \mathbb{Z} .

Definition 3.1. Let $H^{\geq i}(B)H^*(E)$ denote the subgroup in $H^*(E)$ multiplicatively generated by the pullbacks of $H^{\geq i}(B)$. **The Leray-Serre filtration** on $H^*(E)$ is $H^*(E) \supset H^{\geq 1}(B)H^*(E) \supset H^{\geq 2}(B)H^*(E) \supset \dots$. This filtration is clearly multiplicative. The associated graded ring

$$\bigoplus_{p,q} E_{\infty}^{p,q}, \text{ where } E_{\infty}^{p,q} := \frac{H^p(B)H^q(E)}{H^{\geq p+1}(B)H^*(E)}$$

is isomorphic to $H^*(E)$ as a vector space (if the base coefficients are a field); however, the ring structure is often different. **The Leray-Serre spectral sequence** is a collection of spaces $E_i^{p,q}$, with $i \geq 2$ and differentials $d_i : E_i^{p,q} \rightarrow E_i^{p+i, q-i+1}$, $d_i^2 = 0$, with the following properties.

- $E_2^{p,q} = H^p(B) \otimes H^q(F)$, considered as a ring. For each i , the space $\bigoplus_{p,q} E_i^{p,q}$ is a graded commutative ring.
- The differentials d_i acting on $\bigoplus_{p,q} E_i^{p,q}$ satisfy the Leibnitz rule, and $E_{i+1}^{p,q}$ is equal to the cohomology algebra of $(\bigoplus_{p,q} E_i^{p,q}, d_i)$.
- $E_{\infty}^{p,q} = \lim_i E_i^{p,q}$ (this is expressed by saying “the spectral sequence converges to $E_{\infty}^{p,q}$ ”)

The ring $\bigoplus_{p,q} E_k^{p,q}$ is called **the k -th page of the spectral sequence**. If $d_i = 0$ for $i \geq k$, we say that the spectral sequence **degenerates at the page k** .

Exercise 3.1. Let $\pi : E \rightarrow B$ be a fibration with the fiber a torus. Assume that $d_2 = 0$. Prove that all differentials d_i vanish.

Exercise 3.2. Let $\pi : E \rightarrow B$ be a fibration with the fiber a torus. Assume that the pullback map $\pi^* : H^2(B) \rightarrow H^2(E)$ is injective. Prove that all differentials d_i vanish.

Hint. Use the previous exercise.

Exercise 3.3. Let $\pi : E \rightarrow B$ be a fibration with the fiber a complex projective space. Prove that $d_2 = 0$. Prove that all differentials d_i vanish if $d_3 = 0$.

Exercise 3.4. Let $\tau : F \rightarrow E$ be the standard embedding map. Prove that the sequence

$$0 \rightarrow H^1(B) \xrightarrow{\pi^*} H^1(E) \xrightarrow{\tau^*} H^1(F) \xrightarrow{d_2} H^2(B) \xrightarrow{\pi^*} H^2(E)$$

is exact.

Exercise 3.5. Let $F = S^k$, that is, $\pi : E \rightarrow B$ is a sphere bundle. Prove that all differentials except d_{k+1} vanish. Construct **the Gysin exact sequence**

$$\dots \rightarrow H^p(B) \rightarrow H^{p+k+1}(B) \xrightarrow{\pi^*} H^{p+k+1}(E) \rightarrow H^{p+1}(B) \rightarrow \dots$$

Hint. Note that $E_2^{p,q} = H^p(B)$ for $q = 0, k$ and $E_2^{p,q} = 0$ otherwise.

Exercise 3.6. Let $\pi : E \rightarrow B$ be a fibration with E contractible. Assume that F a $(2q - 1)$ -dimensional sphere. Prove that $H^*(B)$ is a free polynomial algebra with one generator in dimension $2q$.

Hint. Use the Gysin sequence and apply multiplicativity of the spectral sequence.

Exercise 3.7. Let $\pi : E \rightarrow B$ be a fibration with $B = S^k$. Prove that all differentials except d_k vanish. Construct an exact sequence

$$\dots \rightarrow H^{p+k}(F) \xrightarrow{\tilde{d}_k} H^p(F) \xrightarrow{\mu} H^{p+k}(E) \rightarrow H^{p+k+1}(F) \rightarrow \dots$$

where μ is multiplication by $\pi^* \text{Vol}_{S^d}$, and \tilde{d}_k is equal to d_k after the identification $H^p(F) = H^k(S^k) \otimes H^p(F) = E_2^{k,p}$.

Exercise 3.8. Let $\pi : E \rightarrow B$ be a fibration with the fiber F an odd-dimensional sphere. Assume that the rings $H^*(B)$ and $H^*(E)$ are freely generated by generators in odd degrees. Prove that all d_i vanish.

Exercise 3.9. Let $\pi : E \rightarrow B$ be a fibration with E contractible, and $H^*(B)$ and $H^*(F)$ free graded algebras, with W^* the space of generators of $H^*(F)$. Prove that $d_i \Big|_{W^{i-1}} \rightarrow H^i(B)$ is injective, and $\bigoplus_i d_i(W^{i-1})$ freely generates $H^*(B)$.