Seiberg-Witten invariants,

lecture 1: Clifford algebras and the Spin group

IMPA, sala 236

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Clifford algebras

DEFINITION: The Clifford algebra of a vector space V with a scalar product q is an algebra generated by V with a relation $xy + yx = -q(x, y)1$, that is, a quotient of $T^{\otimes}V := k \oplus V \oplus V \otimes V \oplus ... \oplus T^{\otimes i}V \oplus ...$ by an ideal

$$T^{\otimes}V \cdot (xy + yx + q(x, y)) \cdot T^{\otimes}V$$

generated by the relation $xy + yx = -q(x, y)$ for all $x, y \in V$.

EXAMPLE: If $g = 0$, Clifford algebra is Grassmann algebra.

EXAMPLE: $\mathrm{Cl}(\mathbb{R}, +) = \mathbb{C}$ **(prove it)**.

EXAMPLE: $\mathrm{Cl}(\mathbb{R}, -) = \mathbb{R} \oplus \mathbb{R}$ **(prove it)**.

EXAMPLE: $\mathrm{Cl}(\mathbb{R}^2, ++) = \mathbb{H}$ **(prove it)**.

EXAMPLE: $\mathrm{Cl}(\mathbb{R}^2, --) = \mathrm{Cl}(\mathbb{R}^2, +-) = \mathrm{Mat}(2, \mathbb{R})$ **(left as a homework)**.

Grading on the Clifford algebra

DEFINITION: A $\mathbb{Z}/2\mathbb{Z}$-graded algebra is an algebra $A = A_{\text{even}} \oplus A_{\text{odd}}$ such that $A_{\text{even}}A_{\text{odd}} \subset A_{\text{odd}}$, $A_{\text{odd}}A_{\text{even}} \subset A_{\text{odd}}$, $A_{\text{even}}A_{\text{even}} \subset A_{\text{even}}$ and $A_{\text{odd}}A_{\text{odd}} \subset A_{\text{even}}$.

EXAMPLE: The Clifford algebra is $\mathbb{Z}/2\mathbb{Z}$ graded: $\mathrm{Cl}(V,g) = \mathrm{Cl}_{\text{even}}(V,g) \oplus \mathrm{Cl}_{\text{odd}}(V,g)$, with $\mathrm{Cl}_{\text{odd}}(V,g)$ generated by products of odd number of $x_i \in V$, and $\mathrm{Cl}_{\text{even}}(V,g)$ by products of even number of x_i.

EXAMPLE: The Grassmann algebra $\Lambda^*(V) = \mathrm{Cl}(V,0)$ is $\mathbb{Z}/2\mathbb{Z}$ graded (odd forms are odd, even forms are even). This is a special case of the previous example.

DEFINITION: Let $A = A_{\text{even}} \oplus A_{\text{odd}}$ be a graded associative algebra. Let A^{sign} be the same vector space with the new multiplication $a \bullet a' := (-1)^{\tilde{a}\tilde{a}'} aa'$.

EXERCISE: Prove that A^{sign} is associative. Construct an isomorphism $\Lambda^*(V) \cong \Lambda^*(V)^{\text{sign}}$

EXERCISE: Prove that $\mathrm{Cl}(V,g)^{\text{sign}} = \mathrm{Cl}(V,-g)$.

EXERCISE: Define G-graded algebras, where G is any group or semigroup.

Associated graded algebra

DEFINITION: Let R be an associative algebra. **Filtration** on R is a collection of subspaces $R_0 \subset R_1 \subset R_2 \subset ...$ such that $R_i R_j \subset R_{i+j}$.

REMARK: Let $x \in R_k, y \in R_l$. Then the product xy modulo R_{k+l-1} depends only on the class of x modulo R_{k-1}. Indeed, $R_{k-1} R_l \subset R_{k+l-1}$. **This defines the product map $(R_k/R_{k-1}) \otimes (R_l/R_{l-1}) \longrightarrow R_{k+l}/R_{k+l-1}$. We obtained the associative product structure on the space $\bigoplus_{i=0}^{\infty} R_i/R_{i-1}$.**

DEFINITION: Let $R_0 \subset R_1 \subset R_2 \subset ...$ be a filtered algebra. The algebra $\bigoplus_{i=0}^{\infty} R_i/R_{i-1}$ is called **the associated graded algebra** of this filtration.

EXAMPLE: Let $\operatorname{Diff}^k(M)$ be the space of differential operators of order $\leqslant k$ on a smooth manifold M. Consider the filtration $\operatorname{Diff}^0(M) \subset \operatorname{Diff}^1(M) \subset \operatorname{Diff}^2(M) \subset ...$ on the ring of differential operators. The associated graded ring is called **the ring of symbols of differential operators**.

EXERCISE: Prove that the ring of symbols **is isomorphic to the space $\operatorname{Sym}^*(TM)$ of symmetric polyvectors.**

Associated graded algebra of the Clifford algebra

THEOREM: Define a filtration $F^i \mathsf{Cl}(V)$ by induction, $F^0\mathsf{Cl}(V) = \langle 1\rangle$, $F^1\mathsf{Cl}(V) = \langle 1+V\rangle$, $F^2\mathsf{Cl}(V) = \langle 1+V+V\cdot V\rangle$, ..., $F^{i+1}\mathsf{Cl}(V) = F^i\mathsf{Cl}(V) + F^i\mathsf{Cl}(V)\cdot V$. **Then the associated graded algebra of $\mathsf{Cl}(V)$ is isomorphic to the Grassmann algebra.**

Proof. Step 1: Let $F^i(T^\otimes V) := k \oplus V \oplus V\otimes V \oplus ... \oplus T^{\otimes i}V$. Clearly, this filtration is compatible with the filtration on the Clifford algebra. This implies that

$$\frac{F^{i+1}\mathsf{Cl}(V)}{F^i\mathsf{Cl}(V)} = \frac{F^{i+1}T^\otimes V}{F^iT^\otimes V + F^{i+1}T^\otimes V\cap I}, \quad (*)$$

where $I := T^\otimes V \otimes (x\otimes y + y\otimes x + q(x,y))\otimes T^\otimes V$ is the ideal of relations defining the Clifford algebra.

Step 2: Let $I_0 := T^\otimes V\otimes(x\otimes y+y\otimes x)\otimes T^\otimes V$. This is the ideal of relations in the Grassmann algebra. Clearly, $F^iT^\otimes V + F^{i+1}T^\otimes V\cap I = F^iT^\otimes V + F^{i+1}T^\otimes V\cap I_0$. Then (*) implies that the associated graded algebra of $\mathsf{Cl}(V)$ is

$$\frac{F^{i+1}\mathsf{Cl}(V)}{F^i\mathsf{Cl}(V)} = \frac{F^{i+1}T^\otimes V}{F^iT^\otimes V + F^{i+1}T^\otimes V\cap I_0},$$

which is identified with the associated graded of the Grassmann algebra. Since the relations in the Grassmann algebra is homogeneous, **the latter is isomorphic to the Grassmann algebra itself.** ■

COROLLARY: $\dim \mathsf{Cl}(V) = 2^{\dim V}$. ■

Graded tensor product

DEFINITION: Let $A := A_{\text{even}} \oplus A_{\text{odd}}$, $B := B_{\text{even}} \oplus B_{\text{odd}}$ be graded associative algebras. Define **the graded tensor product** $A \tilde{\otimes} B$ as $A \otimes B$ with multiplication given by $a \otimes b \cdot a' \otimes b' = (-1)^{\tilde{b}\tilde{a}'} aa' \otimes bb'$, where $\tilde{x}$ denotes the parity of x.

EXAMPLE: Graded tensor product of Grassmann algebras **gives the Grassmann algebra of a direct sum:**

$$\Lambda^* V \tilde{\otimes} \Lambda^* W \cong \Lambda^*(V \oplus W)$$

EXAMPLE: The same is true for Clifford algebras:

$$\mathrm{Cl}(V, g) \tilde{\otimes} \mathrm{Cl}(V', g') = \mathrm{Cl}(V \oplus V', g + g').$$

Pseudoscalar

LEMMA (*): Let $A := A_{\text{even}} \oplus A_{\text{odd}}$, $B := B_{\text{even}} \oplus B_{\text{odd}}$ be graded associative algebras. Suppose that B contains an even element **(pseudoscalar)** ε with the following properties:

$$\varepsilon^2 = 1, \varepsilon b = (-1)^{\tilde{b}} b\varepsilon.$$

Then $A \tilde{\otimes} B \cong A \otimes B$ (the graded tensor product is isomorphic to the usual one).

Proof: Consider a subalgebra $A' \subset A \tilde{\otimes} B$ generated by elements $a \tilde{\otimes} \varepsilon^{\tilde{a}}$ and $B' = 1 \otimes B \subset A \tilde{\otimes} B$. Then

1. $A' \cong A$ **commutes with** $B' \cong B$.

2. $A' \otimes B' = A \tilde{\otimes} B$ **as a vector space.** ■

REMARK (*): If in the definition of pseudoscalar we replace $\varepsilon^2 = 1$ by $\varepsilon^2 = -1$, **Lemma (*) will give** $A \tilde{\otimes} B \cong A^{\text{sign}} \otimes B$.

Unit pseudoscalar

DEFINITION: Let (V, g) be an oriented real vector space with orthogonal basis $e_1, ..., e_n$ such that $g(e_i, e_i) = \pm 1$. **Unit pseudoscalar** in $\mathsf{Cl}(V, g)$ is $\varepsilon := e_1 e_2 e_3 ... e_n$.

EXERCISE: Prove $\varepsilon e_i = (-1)^{n-1} e_i \varepsilon$. In other words, **$\varepsilon$ is a pseudoscalar when n is even.**

EXERCISE: Prove that $\varepsilon^2 = (-1)^{\frac{(n)(n-1)}{2}}(-1)^q$ **if g has signature (p,q).**

REMARK: Since $(p+q)^2 = (p-q)^2 \mod 4$, we have $(p+q)(p+q-1)+2q = (p+q)^2 + 2q - (p+q) = (p-q)^2 - (p-q) = (p-q)(p-q-1) \mod 4$. This gives

$$\varepsilon^2 = (-1)^{(p+q-1)(p+q-2)/2+q} = (-1)^{(p-q)(p-q-1)/2} = \begin{cases} +1 & p-q \equiv 0,1 \mod 4 \\ -1 & p-q \equiv 2,3 \mod 4. \end{cases}$$

COROLLARY: Denote **the Clifford algebra of a real vector space of signature (p,q) by $\mathsf{Cl}(p,q)$.** Then $\mathsf{Cl}(p+m, q+m') \cong \mathsf{Cl}(p,q) \otimes \mathsf{Cl}(m,m')$ **when $m+m'$ is even, and $m-m' \equiv 0 \mod 4$.**

Proof: The pseudoscalar ε in $\mathsf{Cl}(m,m')$ satisfies $\varepsilon^2 = 1$. Applying Lemma (*), we obtain $\mathsf{Cl}(p,q) \otimes \mathsf{Cl}(m,m') \cong \mathsf{Cl}(p,q) \tilde{\otimes} \mathsf{Cl}(m,m')$. Then we apply the isomorphism $\mathsf{Cl}(V,g) \tilde{\otimes} \mathsf{Cl}(V',g') = \mathsf{Cl}(V \oplus V', g+g')$. ■

Bott periodicity over $\mathbb{C}$

COROLLARY: Let $A[i]$ denote the tensor product $A \otimes \mathrm{Mat}(i) \cong \mathrm{Mat}(i, A)$. **Then** $\mathrm{Cl}(p+1, q+1) \cong \mathrm{Cl}(p,q)[2]$**.**

Proof: Use the previous corollary and an isomorphism $\mathrm{Cl}(1,1) = \mathrm{Mat}(2, \mathbb{R})$ **(prove it).** ■

THEOREM: (Bott periodicity over $\mathbb{C}$)
Clifford algebra $\mathrm{Cl}(V, q)$ of a complex vector space $V = \mathbb{C}^n$ with q non-degenerate **is isomorphic to** $\mathrm{Mat}\left((\mathbb{C})^{2^{n/2}}\right)$ **(n even) and** $\mathrm{Mat}\left(\mathbb{C}^{2^{\frac{n-1}{2}}}\right) \oplus \mathrm{Mat}\left(\mathbb{C}^{2^{\frac{n-1}{2}}}\right)$ **(n odd).**

Proof: Use the previous corollary and isomorphisms $\mathrm{Cl}(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$, $\mathrm{Cl}(0) = \mathbb{C}$. ■

Bott periodicity over $\mathbb{R}$.

CLAIM: $\mathrm{Cl}(p+m, q+m') \cong \mathrm{Cl}(q,p) \otimes \mathrm{Cl}(m,m')$ **if** $m+m'$ **is even, and** $m-m' \equiv 2 \mod 4$.

Proof: In $\mathrm{Cl}(m,m')$ the pseudoscalar ε satisfies $\varepsilon^2 = -1$. Applying Remark (*), we obtain $\mathrm{Cl}(p,q)^{\mathrm{sign}} \otimes \mathrm{Cl}(m,m') \cong \mathrm{Cl}(p,q) \tilde{\otimes} \mathrm{Cl}(m,m') \cong \mathrm{Cl}(p+m, q+m')$. Then we use an isomorphism $\mathrm{Cl}(p,q)^{\mathrm{sign}} = \mathrm{Cl}(p,q)$. ■

COROLLARY: $\mathrm{Cl}(p+2,q) \cong \mathrm{Cl}(q,p)[2]$ **and** $\mathrm{Cl}(p,q+2) \cong \mathrm{Cl}(q,p) \otimes \mathbb{H}$.

Proof: We use the previous claim and the isomorphisms $\mathrm{Cl}(2,0) = \mathrm{Mat}(2,\mathbb{R})$, $\mathrm{Cl}(0,2) = \mathbb{H}$. ■

COROLLARY: (Bott Periodicity modulo 4):
The previous corollary **immediately gives** $\mathrm{Cl}(p+4,q) \cong \mathrm{Cl}(q,p+2)[2] = \mathrm{Cl}(p,q) \otimes \mathrm{Mat}(2,\mathbb{H})$ **and** $\mathrm{Cl}(p,q+4) \cong \mathrm{Cl}(q+2,p) \otimes \mathbb{H} = \mathrm{Cl}(p,q) \otimes \mathrm{Mat}(2,\mathbb{H})$.

Bott periodicity modulo 8.

EXERCISE: Prove the isomorphism $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} = \operatorname{Mat}(4, \mathbb{R})$.

COROLLARY: (Bott Periodicity modulo 8):
This isomorphism and the previous corollary give $\operatorname{Cl}(p+8, q) = \operatorname{Cl}(p, q)[16]$, $\operatorname{Cl}(p, q+8) = \operatorname{Cl}(p, q)[16]$.

The table of Clifford algebras $\operatorname{Cl}(p, q)$

$p+q$ \ $q-p$	8	7	6	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8
0									$\mathbb{R}$								
1								$\mathbb{R}^2$		$\mathbb{C}$							
2							$M(2,\mathbb{R})$		$M(2,\mathbb{R})$		$\mathbb{H}$						
3						$M(2,\mathbb{C})$		$M(2,\mathbb{R})^2$		$M(2,\mathbb{C})$		$\mathbb{H}^2$					
4					$M(2,\mathbb{H})$		$M(4,\mathbb{R})$		$M(4,\mathbb{R})$		$M(2,\mathbb{H})$		$M(2,\mathbb{H})$				
5				$M(2,\mathbb{H})^2$		$M(4,\mathbb{C})$		$M(4,\mathbb{R})^2$		$M(4,\mathbb{C})$		$M(2,\mathbb{H})^2$		$M(4,\mathbb{C})$			
6			$M(4,\mathbb{H})$		$M(4,\mathbb{H})$		$M(8,\mathbb{R})$		$M(8,\mathbb{R})$		$M(4,\mathbb{H})$		$M(4,\mathbb{H})$		$M(8,\mathbb{R})$		
7		$M(8,\mathbb{C})$		$M(4,\mathbb{H})^2$		$M(8,\mathbb{C})$		$M(8,\mathbb{R})^2$		$M(8,\mathbb{C})$		$M(4,\mathbb{H})^2$		$M(8,\mathbb{C})$		$M(8,\mathbb{R})^2$	
8	$M(16,\mathbb{R})$		$M(8,\mathbb{H})$		$M(8,\mathbb{H})$		$M(16,\mathbb{R})$		$M(16,\mathbb{R})$		$M(8,\mathbb{H})$		$M(8,\mathbb{H})$		$M(16,\mathbb{R})$		$M(16,\mathbb{R})$
ε^2	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+	+

Pseudoscalar on an odd-dimensional space

For any odd-dimensional space V, the pseudoscalar $\varepsilon = e_1 e_2 ... e_{2n+1}$ commutes with a multiplication by generators of $\mathrm{Cl}(V)$, hence defines an automorphism of $\mathrm{Cl}(V)$. If V were a complex vector space, we can always chose the basis $e_1, e_2, ..., e_{2n+1}$ in such a way that $\varepsilon^2 = 1$. Indeed,

$$\varepsilon^2 = (-1)^{(p+q-1)(p+q-2)/2+q} = (-1)^{(p-q)(p-q-1)/2} = \begin{cases} +1 & p-q \equiv 0,1 \mod 4 \\ -1 & p-q \equiv 2,3 \mod 4. \end{cases}$$

This pseudoscalar is central, because V is odd-dimensional. **This gives the eigenvalue decomposition $\mathrm{Cl}(V) = \mathrm{Cl}^+(V) \oplus \mathrm{Cl}^-(V)$.**

CLAIM: Each of the algebras $\mathrm{Cl}^+(V)$, $\mathrm{Cl}^-(V)$ is isomorphic to $\mathrm{Mat}(\mathbb{C}^r)$.

Proof: Eigenvalues of ε acting on $\mathrm{Cl}(V)$ are equal to ± 1 because $\varepsilon^2 = 1$. On the other hand, an automorphism of V which exchanges e_1 and e_2 maps ε to $-\varepsilon$, hence permutes the eigenspaces. Therefore, **the subalgebras $\mathrm{Cl}^+(V)$, $\mathrm{Cl}^-(V)$ are isomorphic.** We obtain that **the decomposition $\mathrm{Cl}(V) = \mathrm{Mat}(2^n, \mathbb{C}) \oplus \mathrm{Mat}(2^n, \mathbb{C})$ coincides with the eigenspace decomposition defined by ε.**

REMARK: The center of $\mathrm{Cl}(V)$ is isomorphic to $\mathbb{C} \oplus \mathbb{C}$. **The orthogonal group $O(V)$ acts on $\mathrm{Cl}(V)$ by automorphisms, and maps ε to $\pm\varepsilon$.** In particular, **$SO(V)$ acts on $\mathrm{Cl}^{\pm}(V)$ by automorphisms**.

Spinorial group Spin$(2n)$

EXERCISE: Let V be a vector space over a field of characteristic 0. Prove that **the automorphism group** Aut(Mat(V)) **is isomorphic to** $PGL(V)$ (the quotient of $GL(V)$ by its center).

REMARK: Let $V = \mathbb{C}^{2n}$ be a vector space over $\mathbb{C}$ with non-degenerate scalar product. **The group $SO(V)$ acts on Cl(V) by automorphisms,** giving an action $SO(V) \hookrightarrow \operatorname{Aut}(\operatorname{Mat}(2^n, \mathbb{C})) = PGL(2^n, \mathbb{C})$, as shown above.

DEFINITION: Clifford module is a faithful representation of the Clifford algebra; when the Clifford algebra is Mat(A, n), its Clifford modules are isomorphic to A^{kn}.

DEFINITION: (Elie Cartan, 1913)
Spinor representation of the Lie algebra $\mathfrak{so}(V)$ is its representation on the Clifford module $\mathbb{C}^{2^n}$ induced by the isomorphism $\mathfrak{pgl}(2^n) = \mathfrak{sl}(2^n)$.

EXERCISE: Prove that $\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z}$, for $n \geqslant 3$.

DEFINITION: Spinor group Spin$(2n)$ is the double cover of $SO(2n)$ obtained as a Lie group of $\mathfrak{so}(V)$ acting on its spinor representation.

Spinorial group Spin$(2n+1)$

REMARK: Let $V = \mathbb{C}^{2n+1}$ be a vector space over $\mathbb{C}$ with non-degenerate scalar product. The group $SO(V)$ acts on $\mathrm{Cl}(V)$ be automorphisms, and defines a homomorphism

$$SO(V) \hookrightarrow \mathrm{Aut}(\mathrm{Mat}(2^n, \mathbb{C}) \oplus \mathrm{Mat}(2^n, \mathbb{C})) = PGL(2^n, \mathbb{C}) \times PGL(2^n, \mathbb{C}).$$

DEFINITION: (Elie Cartan, 1913)
Spinor representation of the Lie algebra $\mathfrak{so}(V)$, $V = \mathbb{C}^{2n+1}$, is its representation on the Clifford module $\mathbb{C}^{2^n}$ induced by the isomorphism $\mathfrak{pgl}(2^n, \mathbb{C}) = \mathfrak{sl}(2^n, \mathbb{C})$.

DEFINITION: Spinor group Spin$(2n+1)$ is a double cover of $SO(2n+1)$ obtained as a Lie group of $\mathfrak{so}(V)$ acting on its spinorial representation.

EXERCISE: In even- and odd-dimensional case, prove that Spin(r) **is, indeed, a double cover of $SO(r)$.**

REMARK: Later today we will see that the Spin group **is embedded as a multiplicative subgroup in the Clifford algebra, and acts faithfully in each Clifford module.**

An isomorphism $\mathfrak{so}(V) = \Lambda^2(V)$

CLAIM: Let g be a non-degenerate scalar product on V. Then the Lie algebra $\mathfrak{so}(V)$ **is naturally isomorphic to $\Lambda^2(V)$.**

Proof: Consider g as a map $\hat{g}: V \to V^*$, and let $\Psi: V \otimes V \to \operatorname{End}(V)$ take $x \otimes y$ to $x \otimes \hat{\mathfrak{g}}(y) \in V \otimes V^* = \operatorname{End}(V)$. Then for any $v \in V$, we have $\psi(\eta)(v) = i_v(\eta)$, where i_v is the contraction with the 1-form $g(v)$. Then for any $\eta \in V \otimes V$, we have $g(\Psi(\eta)(x), y) = g(i_x(\eta), y) = i_x i_y(\eta)$. Clearly, **this map is skew-symmetric if and only if $\eta \in \Lambda^2 V$.** ■

$\mathfrak{so}(V)$-action on $\mathrm{Cl}(V)$

Remark 1: Let $\alpha_1, \alpha_2, ...$ be derivations acting on an associative algebra A and preserving its set of generators W. Then $[\alpha_i, \alpha_j]$ is also a derivation, and its action on A is extended from W. In particular, **all Lie relations between α_i which are true on W remains true on A.**

CLAIM: Consider the map $V \otimes V \to \mathrm{Cl}(V)$ taking $x \otimes y$ to $xy + g(x,y)$. Clifford identities $xy + g(x,y) = -yx - g(x,y)$ imply that this map is antisymmetric, hence defines a linear operator $\Psi : \Lambda^2(V) \to \mathrm{Cl}(V)$. **Then its image is a Lie subalgebra of $\mathrm{Cl}(V)$.** Moreover, it is isomorphic to $\mathfrak{so}(V)$

Proof. Step 1: Let $x, y \in V$ be orthogonal vectors. Then $xy = -yx$, by Clifford identities, hence $\Psi(x \otimes y) = xy$. Clearly, $xyt - txy = xg(y,t) - yg(x,y)$, hence $a \to [xy, a]$ **acts on $V \subset \mathrm{Cl}(V)$ as an element of $\mathfrak{so}(V)$ corresponding to $x \wedge y$.**

Step 2: The operation $a \to [xy, a]$ **is a derivation,** hence the $\mathfrak{so}(V)$-action on V defined above extends to the action of the same Lie algebra on $\mathrm{Cl}(V)$ (Remark 1). ■

Spin group as a subgroup in the group of invertible elements of $\mathrm{Cl}(V)$

REMARK: For any element $x \in \mathrm{Cl}(V)$, we define $\exp(x) := \sum_i \frac{x^i}{i!}$. Clearly, $\exp(x)$ acts on $\mathrm{Cl}(V)$ as matrix exponents of the map $v \to xv$. **T herefore, for any Lie subalgebra $\mathfrak{g} \subset \mathrm{Der}(\mathrm{Cl}(V))$ of derivations of $\mathrm{Cl}(V)$, its exponent acts on $\mathrm{Cl}(V)$ by automorphisms.** Moreover, $\exp(\mathfrak{g})$ **generates a Lie group G which satisfies $\mathrm{Lie}(G) = \mathfrak{g}$.**

THEOREM: Consider the Lie algebra embedding $\mathfrak{so}(V) \to \mathrm{Der}(\mathrm{Cl}(V))$ constructed above. **Then its exponent is the Spin group acting on $\mathrm{Cl}(V)$ by automorphisms.**

Proof. Step 1: Clearly, an exponent of a derivation is an automorphism, and the exponent of $\mathfrak{so}(V)$ is $SO(V)$ or its covering. **It remains to show that this subgroup is the Spin group, and not its quotient $SO(V)$.**

Step 2: Since $\mathrm{Cl}(\mathbb{R}^3, +++) = \mathbb{H}^2$, and its spinorial representation is $\mathbb{H}$. **This gives $\mathrm{Spin}(3) = SU(2)$, where $SU(2) = U(1, \mathbb{H})$ is the group of unit quaternions. This proves the claim when $V = \mathbb{R}^3$.**

Step 3: For any $V = \mathbb{R}^n$, $n > 3$, there is a 3-dimensional subspace $W \subset V$, and **the corresponding subgroup $G(V) \subset \mathrm{Cl}(V)$ is isomorphic to $SU(2)$,** while its Lie algebra is isomorphic to $\mathfrak{so}(3)$. This is impossible if $G(V) = SO(V)$ hence $G(V)$ is the 2-sheeted covering of $SO(V)$. ■

REMARK: By construction, for any $\eta \in \Lambda^2(V)$, the element of $\mathrm{Spin}(V)$ corresponding to η is the map $x \to e^\eta x e^{-\eta}$. **Therefore, $\mathrm{Spin}(V) \subset \mathrm{Cl}(V)^*$ is the multiplicative subgroup of $\mathrm{Cl}(V)$ generated by e^η, for all $\eta \in \Lambda^2(V)$.**