Seiberg-Witten invariants,

lecture 2: Spin structures

IMPA, sala 236

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Clifford algebras (reminder)

DEFINITION: The Clifford algebra of a vector space V with a scalar product q is an algebra generated by V with a relation $xy + yx = -q(x, y)1$, that is, a quotient of $T^{\otimes}V := k \oplus V \oplus V \otimes V \oplus ... \oplus T^{\otimes i}V \oplus ...$ by an ideal

$$T^{\otimes}V \cdot (xy + yx + q(x, y)) \cdot T^{\otimes}V$$

generated by the relation $xy + yx = -q(x, y)$ for all $x, y \in V$.

EXAMPLE: If $q = 0$, Clifford algebra is Grassmann algebra.

THEOREM: Let $V = \mathbb{C}^n$ with non-degenerate scalar product. **Then $\mathrm{Cl}(V) = \mathrm{Mat}(\mathbb{C}, 2^{n/2})$ for even n and $\mathrm{Cl}(V) = \mathrm{Mat}(\mathbb{C}, 2^{n-1/2}) \oplus \mathrm{Mat}(\mathbb{C}, 2^{n-1/2})$ for odd n.**

Proof: Lecture 1. ■

$\mathrm{Cl}(W \oplus W^*) = \mathrm{Mat}(\Lambda^* W)$

THEOREM: Let $V := W \oplus W^*$, with the usual pairing $\langle (x+\xi), (x'+\xi')\rangle = \xi(x') + \xi'(x)$. **Then $\mathrm{Cl}(V)$ is naturally isomorphic to $\mathrm{Mat}(\Lambda^* V^*)$.**

Proof. Step 1: Consider the convolution map $W \otimes \Lambda^i W^* \longrightarrow \Lambda^{i-1} W^*$, with $v \otimes \xi \longrightarrow \xi(v, \cdot, \cdot, \ldots, \cdot)$ denoted by $v, \xi \longrightarrow i_v(\xi)$ and the extertior multiplication map $W^* \otimes \Lambda^i W^* \longrightarrow \Lambda^{i-1} W^*$, with $\nu \otimes \xi \longrightarrow \nu \wedge \xi$, denoted by $\nu, \xi \longrightarrow e_\nu(\xi)$. Let $V \otimes \Lambda^* W^* \xrightarrow{\Gamma} \Lambda^* W^*$ map $(v, \nu) \otimes \xi$ to $i_v(\xi) + e_\nu(\xi)$. Clearly, all i_v pairwise anticommute, all e_ν pairwise anticommute, and the anticommutator $\{i_v, e_\nu\}$ is a scalar operator of multiplication by a number $\nu(v)$.

To prove the last assertion without any calculations, we notice that i_v **is an odd derivation of the Grassmann algebra, e_ν is a linear operator, and a commutator of a derivation and a linear operator is linear,** hence one has

$$\{i_v, e_\nu\}(a) = \{i_v, e_\nu\}(1) \wedge a = \nu(v) \cdot a.$$

Step 2: These relation imply that the map $V \otimes \Lambda^* W^* \xrightarrow{\Gamma} \Lambda^* W^*$, is extended to a homomorphism $\mathrm{Cl}(V) \longrightarrow \mathrm{Mat}(\Lambda^* W^*)$. I's not hard to show that this map is surjective. **Since $\dim \mathrm{Cl}(V) = 2^{\dim V} = 2^{2 \dim W} = \dim \mathrm{Mat}(\Lambda^* W^*)$, this also implies that $\mathrm{Cl}(V) \cong \mathrm{Mat}(\Lambda^* W^*)$.** ■

An isomorphism $\mathfrak{so}(V) = \Lambda^2(V)$

CLAIM: Let q be a non-degenerate scalar product on V. Then the Lie algebra $\mathfrak{so}(V)$ **is naturally isomorphic to $\Lambda^2(V)$.**

Proof: Consider q as a map $\hat{g}: V \to V^*$, and let $\Psi: V \otimes V \to \operatorname{End}(V)$ take $x \otimes y$ to $x \otimes \hat{\mathfrak{g}}(y) \in V \otimes V^* = \operatorname{End}(V)$. Then for any $v \in V$, we have $\psi(\eta)(v) = i_v(\eta)$, where i_v is the contraction with the 1-form $q(v)$. Then for any $\eta \in V \otimes V$, we have $q(\Psi(\eta)(x), y) = q(i_x(\eta), y) = i_x i_y(\eta)$. Clearly, **this map is skew-symmetric if and only if $\eta \in \Lambda^2 V$.** ■

$\mathfrak{so}(V)$-action on $\mathsf{Cl}(V)$

Remark 1: Let $\alpha_1, \alpha_2, ...$ be derivations acting on an associative algebra A and preserving its set of generators W. Then $[\alpha_i, \alpha_j]$ is also a derivation, and its action on A is extended from W. In particular, **all Lie relations between α_i which are true on W remains true on A.**

REMARK: Let $a \in A$ be an element of an associative algebra. Then the map $x \to [a, x]$ **is always a derivation on A.**
Proof: $[a,x]y + x[a,y] = axy - xay + xay - xya = axy - xya = [a, xy]$. ■

CLAIM: Consider the map $V \otimes V \to \mathsf{Cl}(V)$ taking $x \otimes y$ to $xy + q(x,y)$. Clifford identities $xy + q(x,y) = -yx - q(x,y)$ imply that this map is antisymmetric, hence defines a linear operator $\Psi : \Lambda^2(V) \to \mathsf{Cl}(V)$. **Then its image is a Lie subalgebra of $\mathsf{Cl}(V)$.** Moreover, it is isomorphic to $\mathfrak{so}(V)$

Proof. Step 1: Let $x, y \in V$ be orthogonal vectors. Then $xy = -yx$, by Clifford identities, hence $\Psi(x \otimes y) = xy$. Clearly, $xyt - txy = xq(y,t) - yq(x,y)$, hence $a \to [xy, a]$ **acts on $V \subset \mathsf{Cl}(V)$ as an element of $\mathfrak{so}(V)$ corresponding to $x \wedge y$.**

Step 2: The operation $a \to [xy, a]$ **is a derivation,** hence the $\mathfrak{so}(V)$-action on V defined above extends to the action of the same Lie algebra on $\mathsf{Cl}(V)$ (Remark 1). ■

$\mathfrak{so}(V)$ in $\mathsf{Cl}(V)$

CLAIM: The image of the map $\Psi: \Lambda^2(V) \to \mathsf{Cl}(V)$ **is a Lie subalgebra isomorphic to $\mathfrak{so}(V)$.**

Proof: It suffices to compute $[\Psi(x\wedge y), \Psi(z\wedge t)]$, where $x, y \in V$ and $z, t \in V$ are orthogonal vectors. Let $A := \Psi(x\wedge y)$. Then $[\Psi(x\wedge y), \Psi(z\wedge t)] = A(z)t + zA(t)$, and this is precisely how $\mathfrak{so}(V)$ acts on $\Lambda^2(V)$. ■

Exponent in $\mathsf{Cl}(V)$

REMARK: For any element $z \in \mathsf{Cl}(V)$, we define $\exp(z) := \sum_i \frac{z^i}{i!}$.

CLAIM: For any $\eta \in \Lambda^2 V$, **the map $z \to \exp(\Psi(\eta))x \exp(\Psi(\eta))^{-1}$ acts on $\mathsf{Cl}(V)$ as matrix exponents of the map by $v \mapsto [\Psi(\eta), v]$.**

Proof: Let $z \in \mathrm{End}(V)$ and $A(z)(v) := [z, v]$. The relation $e^z v e^{-z} = e^{A(z)} v$ is true in the tensor algebra, because

$$A(z)(v) = \frac{d}{dt} e^{tz} v e^{-tz} = \frac{d}{dt} e^{tA(z)} v,$$

hence $e^z v e^{-z}$ and $e^{A(z)} v$ are exponents of the same vector field. ∎

REMARK: The exponent of $\Psi(V) \subset \mathsf{Cl}(V)$ is a subgroup of the group $\mathsf{Cl}(V)^*$ of invertible elements of $\mathsf{Cl}(V)$. Previous claim implies that the **conjugation with this group acts on $\mathsf{Cl}(V)$ as the connected component $SO^+(V)$ of $SO(V)$.** We are going to prove that **the group $e^{\Psi(V)} \subset \mathsf{Cl}(V)$ is the double cover of $SO^+(V)$, that is, the Spin group.**

Spin group as a subgroup in the group of invertible elements of $\operatorname{Cl}(V)$

THEOREM: Consider the Lie algebra embedding $\mathfrak{so}(V) \to \operatorname{Der}(\operatorname{Cl}(V))$ constructed above. **Then its exponent is the non-trivial double cover of the connected component of $SO(V)$, that is, $\operatorname{Spin}(V)$.**

Proof. Step 1: Clearly, an exponent of a derivation is an automorphism, and the exponent of $\mathfrak{so}(V)$ is the connected component of $SO(V)$ or its covering. **It remains to show that this subgroup is the Spin group, and not its quotient $SO(V)$.**

Step 2: Since $\operatorname{Cl}(\mathbb{R}^3, +++) = \mathbb{H}^2$, and its spinorial representation is $\mathbb{H}$. **This gives $\operatorname{Spin}(3) = SU(2)$, where $SU(2) = U(1, \mathbb{H})$ is the group of unit quaternions. This proves the claim when $V = \mathbb{R}^3$.**

Step 3: For any $V = \mathbb{R}^n$, $n > 3$, there is a 3-dimensional subspace $W \subset V$, and **the corresponding subgroup $G(V) \subset \operatorname{Cl}(V)$ is isomorphic to $SU(2)$,** while its Lie algebra is isomorphic to $\mathfrak{so}(3)$. This is impossible if $G(V) = SO(V)$ hence $G(V)$ is the 2-sheeted covering of $SO(V)$. ■

REMARK: We have shown that $\operatorname{Spin}(V) \subset \operatorname{Cl}(V)^*$ **is the multiplicative subgroup of $\operatorname{Cl}(V)^*$ generated by e^η, for all $\eta \in \Lambda^2(V)$.**

Principal bundles

DEFINITION: Let G be a Lie group. **Principal G-bundle** over a manifold M is a smooth fibration $P \mapsto M$ with a smooth G-action which acts freely and transitively on fibers.

EXAMPLE: Frame bundle on a smooth n-manifold M is the bundle of all frames (basises) in T_xM, for all $x \in M$. **DEFINITION:** Let $H \longrightarrow G$ be a group homomorphism, and P a principal H-bundle. Then the quotient $P_G := P \times G/H$ (with H acting on both components in a natural way) is called **an associated principal bundle**, and P is called **reduction of the principal G-bundle P_G to the group H**.

DEFINITION: Let G be a Lie group, and $G \longrightarrow GL(n, \mathbb{R})$ a group homomorphism. **A G-structure on a manifold M** is a reduction of the principal frame bundle to G.

DEFINITION: Let G be a Lie group, V its representation, and P a principal G-bundle on M. The quotient $P \times V/G$ is a vector bundle over M, called **the associated vector bundle**.

Spin-structures and spinor bundles

DEFINITION: A spin-structure on an oriented real vector bundle with a scalar product is a reduction of its frame bundle (which is a principal $SO(n)$-fibration) to Spin(n).

REMARK: This happens precisely when the second Stiefel-Whitney class $w_2(B)$ vanishes.

CLAIM: Let $P_{SO(n)}(B)$ be the bundle of orthogonal frames in B. Then the Spin-structures on B **are in bijective correspondence with the 2-sheeted covers $\tilde{P}_{SO(n)}(B) \longrightarrow P_{SO(n)}(B)$ defining the universal cover on the fibers.**
■

REMARK: To prove that $w_2(M)$ is an obstruction to existence of the spin structure, we need to consider an exact sequence

$$0 \longrightarrow H^1(M, \mathbb{Z}/2) \longrightarrow H^1(P_{SO(n)}(M), \mathbb{Z}/2) \longrightarrow H^1(SO(n), \mathbb{Z}/2) \stackrel{\delta}{\longrightarrow} H^2(M, \mathbb{Z}/2)$$

and prove that the image of δ is $w_2(B)$. This would follow if we compare this class with $w_2(B)$ for the universal bundle over $BSO(n)$ and then use its functoriality.

Spinor bundle

DEFINITION: A spin-structure on an oriented n-manifold M is a reduction of its structure group to Spin(n). A manifold is called **spin** if it admits a spin-structure.

REMARK: This happens precisely when the second Stiefel-Whitney class $w_2(M)$ vanishes.

DEFINITION: A bundle of spinors on a spin-manifold M is a vector bundle associated to the principal Spin(n)-bundle and a spin representation.

Exercises

EXERCISE: Let $G \subset \mathsf{Cl}(V)^*$ be the subgroup of all $x \in \mathsf{Cl}(V)^*$ such that $xVx^{-1} \subset V$. **Prove that the connected component of G is $\mathsf{Spin}(V)$.**

EXERCISE: Let Pin group $\mathsf{Pin}(V) \subset \mathsf{Cl}(V)^*$ be the subgroup generated by all $v \in V$ such that $q(v,v) = \pm 1$. **Prove that the set of all even-graded elements in Pin is $\mathsf{Spin}(V)$.**

HINT: Use the previous exercise.

EXERCISE: Let $x_1, ..., x_n$ be an oriented orthonormal basis in V, with $q(x_i, x_i) = \pm 1$. The **volume element** of $\mathsf{Cl}(V)$ is $w := x_1 x_2 ... x_n$. Prove that **the center of $\mathsf{Spin}(V)$ is $\pm 1 \subset \mathsf{Cl}(V)$ when n is odd, and $\{\pm 1, \pm w\}$ when it is even.**