Seiberg-Witten invariants,

lecture 3: Spin^c structures

IMPA, sala 236

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Principal bundles (reminder)

DEFINITION: Let G be a Lie group. **Principal G-bundle** over a manifold M is a smooth fibration $P \mapsto M$ with a smooth G-action which acts freely and transitively on fibers.

EXAMPLE: **Frame bundle** on a smooth n-manifold M is the bundle of all frames (basises) in T_xM, for all $x \in M$. **DEFINITION:** Let $H \longrightarrow G$ be a group homomorphism, and P a principal H-bundle. Then the quotient $P_G := P \times G/H$ (with H acting on both components in a natural way) is called **an associated principal bundle**, and P is called **reduction of the principal G-bundle P_G to the group H**.

DEFINITION: Let G be a Lie group, and $G \longrightarrow GL(n, \mathbb{R})$ a group homomorphism. **A G-structure on a manifold M** is a reduction of the principal frame bundle to G.

DEFINITION: Let G be a Lie group, V its representation, and P a principal G-bundle on M. The quotient $P \times V/G$ is a vector bundle over M, called **the associated vector bundle**.

Spin-structures

DEFINITION: A spin-structure on an oriented real vector bundle with a scalar product is a reduction of its frame bundle $P_{SO(n)}(B)$ (which is a principal $SO(n)$-fibration) to Spin(n). A bundle is **spinnable** if it admits a spin-structure.

CLAIM: Let $P_{SO(n)}(B)$ be the bundle of orthogonal frames in B. Then the Spin-structures on B **are in bijective correspondence with the 2-sheeted covers $\tilde{P}_{SO(n)}(B) \longrightarrow P_{SO(n)}(B)$ such that the preimage of each fiber is its universal cover.** ■

COROLLARY: Let B be a real vector bundle on a connected manifold M. Then **the spin-structures on B are in bijective correspondence with elements in $H^1(P_{SO(n)}(B), \mathbb{Z}/2)$ which are non-trivial on the fibers.**

The second Stiefel-Whitney class

DEFINITION: In these assumptions, consider the exact sequence

$$0 \longrightarrow H^1(M, \mathbb{Z}/2) \longrightarrow H^1(P_{SO(n)}(M), \mathbb{Z}/2) \longrightarrow H^1(SO(n), \mathbb{Z}/2) \xrightarrow{\delta} H^2(M, \mathbb{Z}/2)$$

which is obtained from Serre's spectral sequence. The image of δ in $H^2(M, \mathbb{Z}/2)$ **is equal to the second Stiefel-Whitney class of B,** as shown below.

REMARK: Clearly, δ is compatible with pullbacks of vector bundles, **hence functorial,** in the same way as the usual characteristic classes.

DEFINITION: Let $BSO(n)$ be the classifying space of $SO(n)$; as usual, we identify $BSO(n)$ with the Grassmanian $\mathrm{Gr}(n, \infty)$. Then any rank n vector bundle on M can be obtained as a pullback of the universal bundle on $BSO(n)$ for some continuous map $\varphi_B : M \to BSO(n)$. It is not hard to see that $\pi_2(BSO(n)) = H_2(BSO(n)) = \mathbb{Z}/2$ and $H^2(BSO(n), \mathbb{Z}/2) = \mathbb{Z}/2$. Let α be its generator. **The Stiefel-Whitney class $w_2(B)$ is defined as $\varphi_B^*(\alpha)$.**

THEOREM: $w_2 = \delta$.

Proof: Since both δ and w_2 are functorial, it suffices to show that $\delta(B) = w_2(B)$ when B is the universal bundle on $BSO(n)$. Since the Stiefel manifold $P_{SO(n)}(BSO(n))$ is contractible, the universal bundle is not spinnable, hence δ defines a non-zero element in $H^2(BSO(n), \mathbb{Z}/2) = \mathbb{Z}/2$. By definition, w_2 is also non-trivial, which **implies that $w_2(B) = \delta(B)$ when B is the universal bundle on $BSO(n)$.** ■

Spin-structures on almost complex manifold

DEFINITION: A spin-structure on a manifold M is a spin-structure on its tangent bundle.

REMARK: As shown above, **a manifold M admits a spin-structure if $w_2(M) = 0$, and the set of spin-structures is a equipped with a free, transitive $H^1(M, \mathbb{Z})$-action.**

CLAIM: For an almost complex manifold (M, I), **the second Stiefel-Whitney class $w_2(M)$ is equal to $c_1(M, I)$ modulo 2.**

This claim immediately follows from

CLAIM: Let B be a real vector bundle on M, and I an almost complex structure on B. **Then $c_1(B, I)$ modulo 2 is equal to $w_2(B)$.**

Proof. Step 1: From the splitting principle, we obtain that **it suffices to prove the claim when $\operatorname{rk}_{\mathbb{C}} B = 1$.**

Step 2: Clearly, $U(1) = SO(2)$, and their classifying spaces are both equal to $\mathbb{C}P^\infty$. The group $H^2(\mathbb{C}P^\infty, \mathbb{Z})$ is by definition generated by $c_1(B)$, and $H^2(BSO(2), \mathbb{Z}/2) = H^2(\mathbb{C}P^\infty, \mathbb{Z}/2)$ is generated by $w_2(B)$, where B is the universal bundle. ■

Spin-structures in dimension 4

THEOREM: Let M be a compact, oriented, simply-connected 4-manifold. Then M is spin **if and only if the intersection form in $H^2(M,\mathbb{Z})$ is even** (that is, takes only even values).

Proof. Step 1: It suffices to check that $w_2(TM) = 0$ on all generators of $H_2(M,\mathbb{Z})$. Since $\pi_1(M) = 0$, by Hurewicz theorem, $H_2(M,\mathbb{Z}) = \pi_2(M)$. Therefore, it suffices to show that $w_2(TM) = 0$ on any immersed 2-spheres S.

Step 2: Since $TM|_S = NS \oplus TS$ and $w_2(TS) = 0$, Whitney formula implies that $w_2(TM|_S) = w_2(NS)$. As shown on the previous slide, $w_2(B) = e(b) = c_1(B)$ for any oriented rank 2 real bundle. However, $e(NS)$ is self-intersection of S. **Therefore, the self-intersection any homology class $[S] \in H_2(M,\mathbb{Z})$ is even if and only if M is spin.** ■

Spin^c-group

DEFINITION: Let V be a real vector space with a scalar product, and let $z \in \mathrm{Spin}(V)$ be the non-trivial element in the kernel of the homomorphism $\mathrm{Spin}(V) \to SO(V)$. Then $\mathrm{Spin}^c(V) := \frac{\mathrm{Spin}(V)\times U(1)}{(z,-1)}$ is called **the Spin^c-group.**

REMARK: Let $S(V)$ be the Clifford module, that is, the spinorial representation associated with V. Consider the natural map $\mathrm{Cl}(V) \to U(S(V)\otimes \mathbb{C})$. Then $\mathrm{Spin}^c(V)$ is the image of $\mathrm{Spin}(V) \times U(1)$ in $U(S(V)\otimes \mathbb{C})$. Therefore, **the space $U(S(V)\otimes \mathbb{C})$ admits a faithful action of the group $\mathrm{Spin}^c(V)$.**

DEFINITION: A Spin^c-structure on a bundle B, $\mathrm{rk}(B) = n$, is a reduction of its structure group to $\mathrm{Spin}^c(n)$.

Spin^c-structures

REMARK 1: From the exponential exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0$, **we obtain an isomorphism $H^1(M, S^1) = H^2(M, \mathbb{Z})$.** Let c_1 denote the map $H^1(M, S^1) \to H^2(M, \mathbb{Z})$.

THEOREM: Let M be a manifold, and $R: H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}/2)$ be the reduction mod 2 map. An oriented vector bundle B admits a Spin^c-structure **if and only if $w_2(B) \in \operatorname{im} R$.**

Proof. Step 1: Clearly, Spin$^c(V)$ is a double cover of $SO(V)$. Consider the exact sequence

$$H^1(M, \mathbb{Z}/2) \to H^1(M, \operatorname{Spin}^c(n)) \to H^1(M, SO(n)) \oplus H^1(M, S^1) \xrightarrow{w_2 + c_1} H^2(M, \mathbb{Z}/2).$$

It follows that **a principal Spin$^c(n)$-bundle is a principal $SO(n) \times S^1$-bundle P such that $w_2 + c_1(P) = 0$.**

Step 2: In other words, a principal $SO(n) \times S^1$-bundle is a pair of a principal $SO(n)$-bundle B and a principal S^1-bundle L, such that $w_2(B) + c_1(L) = 0$ mod 2.

Step 3: From Remark 1, we obtain that a principal S^1-bundle is uniquely determined by $c_1(L) \in H^2(M, \mathbb{Z})$. Therefore, a vector bundle B admits a Spin^c-structure **if and only if there exists $c_1(L) \in H^2(M, \mathbb{Z})$ such that $w_2(B) + c_1(L) = 0 \mod 2$.** ■

Spin^c-structures: examples

EXAMPLE: Any bundle B with a Spin-structure admits a canonical Spin^c-structure. Indeed, from a principal Spin-bundle P we can always obtain a principal Spin^c-bundle $P \times S^1/(z, -1)$

PROPOSITION: Any complex vector bundle admits a natural Spin^c-structure, which is defined uniquely up to homotopy.

Proof. Step 1: Any $g \in U(n)$ preserves the real part of the Hermitian form, which gives a natural embedding $j: U(n) \to SO(2n)$. **Consider an injective homomorphism $\tilde{j}: U(n) \hookrightarrow \operatorname{Spin}^c(2n)$** which takes a matrix $g \in U(n), g = e^A$ to $(e^A, e^{\frac{1}{2}\operatorname{Tr} A}) \in \frac{\operatorname{Spin}(2n) \times S^1}{\pm 1}$. This homomorphism is an exponent of the following Lie algebra homomorphism: $a \to j(a) \times \frac{1}{2}\operatorname{Tr}(a)$, taking $\mathfrak{u}(n)$ to $\mathfrak{so}(n) + \mathfrak{u}(1)$.

Step 2: Then, **any principal $U(n)$-bundle P is a reduction of a principal Spin$^c(2n)$-bundle $P \times_{U(n)} \operatorname{Spin}^c(2n)$,** where $U(n)$ acts on the summands as $\operatorname{Id} \times \tilde{j}$ ■