

Seiberg-Witten invariants,

lecture 4: Spin^c structures on 4-manifolds and twistor spaces

IMPA, sala 236

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Principal bundles (reminder)

DEFINITION: Let G be a Lie group. **Principal G -bundle** over a manifold M is a smooth fibration $P \mapsto M$ with a smooth G -action which acts freely and transitively on fibers.

EXAMPLE: Frame bundle on a smooth n -manifold M is the bundle of all frames (bases) in $T_x M$, for all $x \in M$.

DEFINITION: Let $H \rightarrow G$ be a group homomorphism, and P a principal H -bundle. Then the quotient $P_G := P \times G / H$ (with H acting on both components in a natural way) is called **an associated principal bundle**, and P is called **reduction of the principal G -bundle P_G to the group H** .

DEFINITION: Let G be a Lie group, and $G \rightarrow GL(n, \mathbb{R})$ a group homomorphism. **A G -structure on a manifold M** is a reduction of the principal frame bundle to G .

DEFINITION: Let G be a Lie group, V its representation, and P a principal G -bundle on M . The quotient $P \times V / G$ is a vector bundle over M , called **the associated vector bundle**.

Spin^c-group (reminder)

DEFINITION: Let V be a real vector space with a scalar product, and let $z \in \text{Spin}(V)$ be the non-trivial element in the kernel of the homomorphism $\text{Spin}(V) \rightarrow \text{SO}(V)$. Then $\text{Spin}^c(V) := \frac{\text{Spin}(V) \times U(1)}{(z, -1)}$ is called **the Spin^c-group**.

REMARK: Let $S(V)$ be the Clifford module, that is, the spinorial representation associated with V . Consider the natural map $\text{Cl}(V) \rightarrow U(S(V) \otimes \mathbb{C})$. Then $\text{Spin}^c(V)$ is the image of $\text{Spin}(V) \times U(1)$ in $U(S(V) \otimes \mathbb{C})$. Therefore, **the space $U(S(V) \otimes \mathbb{C})$ admits a faithful action of the group $\text{Spin}^c(V)$.**

DEFINITION: A **Spin^c-structure** on a bundle B , $\text{rk}(B) = n$, is a reduction of its structure group to $\text{Spin}^c(n)$.

Spin^c-structures (reminder)

REMARK 1: From the exponential exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow S^1 \rightarrow 0$, we obtain an isomorphism $H^1(M, S^1) = H^2(M, \mathbb{Z})$. Let c_1 denote the map $H^1(M, S^1) \rightarrow H^2(M, \mathbb{Z})$.

THEOREM: Let M be a manifold, and $R : H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}/2)$ be the reduction mod 2 map. An oriented vector bundle B admits a Spin^c-structure **if and only if** $w_2(B) \in \text{im } R$.

Proof. Step 1: Clearly, Spin^c(V) is a double cover of $SO(V)$. Consider the exact sequence

$$H^1(M, \mathbb{Z}/2) \rightarrow H^1(M, \text{Spin}^c(n)) \rightarrow H^1(M, SO(n)) \oplus H^1(M, S^1) \xrightarrow{w_2 + c_1} H^2(M, \mathbb{Z}/2).$$

It follows that **a principal Spin^c(n)-bundle is a principal $SO(n) \times S^1$ -bundle P such that $w_2 + c_1(P) = 0$.**

Step 2: In other words, a principal $SO(n) \times S^1$ -bundle is a pair of a principal $SO(n)$ -bundle B and a principal S^1 -bundle L , such that $w_2(B) + c_1(L) = 0 \pmod{2}$.

Step 3: From Remark 1, we obtain that a principal S^1 -bundle is uniquely determined by $c_1(L) \in H^2(M, \mathbb{Z})$. Therefore, a vector bundle B admits a Spin^c-structure **if and only if there exists $c_1(L) \in H^2(M, \mathbb{Z})$ such that $w_2(B) + c_1(L) = 0 \pmod{2}$.** ■

2-forms on 4-manifolds

CLAIM: Let $*$: $\Lambda^2 M \rightarrow \Lambda^2 M$ be the Hodge $*$ operator on a 4-manifold. **Then $*^2 = 1$, and its $+$ and $-$ -eigenspaces, denoted by $\Lambda^+(M)$ and $\Lambda^-(M)$, have dimension 3.**

Proof: The assertion $*^2 = 1$ is obtained by checking it on coordinate monomials because $*(e_1 \wedge e_2) = e_3 \wedge e_4$. The dimension is found because any isometry which changes the orientation takes $*$ to $-*$, **hence $\Lambda^+(M)$ and $\Lambda^-(M)$ are conjugate.**

CLAIM: Let V be a 4-dimensional real vector space with scalar product. Consider the isomorphism $\Lambda^2 V \cong \mathfrak{so}(V)$. Under this isomorphism, the eigenvalue components in the decomposition $\Lambda^2 V = \Lambda^+(V) \oplus \Lambda^-(V)$ **corresponds to Lie subalgebras, giving a natural decomposition $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.**

Proof. Step 1: Consider the action $U(1, \mathbb{H}) \times U(1, \mathbb{H}) \rightarrow SO(4)$ taking $h_1, h_2 \in \mathbb{H}$ to the map $x \rightarrow h_1 x h_2^{-1} \in \text{End}_{\mathbb{R}}(\mathbb{H})$. Clearly, the kernel of this map is $(-1, -1)$ **(prove this as an exercise)**, and the image is 6-dimensional, hence this map is surjective. **This implies an isomorphism $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$.**

Step 2: The decomposition $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ is the unique $SO(4)$ -invariant decomposition; however, the decomposition $\Lambda^2 V = \Lambda^+(V) \oplus \Lambda^-(V)$ is also $SO(4)$ -invariant. ■

Twistor spaces

DEFINITION: Let M be an oriented Riemannian 2-manifold. Consider the S^2 -fibration $S\Lambda^+(M) \xrightarrow{\pi} M$ of unit spheres in $\Lambda^+(M)$. The total space of this fibration is called **the twistor space of M** , denoted $\text{Tw}(M)$.

DEFINITION: Each unit vector $s \in \Lambda_m^+(M)$ defines a complex structure operator $u(s) \in \text{End}(T_m M)$ as explained above. Using the Levi-Civita connection, we decompose the tangent bundle $T\text{Tw}(M) = T_{\text{vert}}\text{Tw}(M) \oplus T_{\text{hor}}\text{Tw}(M)$, where $T_{\text{vert}}\text{Tw}(M) = \ker D\pi$ is the vertical tangent space, and $T_{\text{hor}}\text{Tw}(M) \cong \pi^*TM$ the complementary bundle obtained from the Ehresmann connection associated with the Levi-Civita connection.

REMARK: For the present purposes, any complementary bundle will work, for instance, **we could pick any Riemannian metric on $\text{Tw}(M)$, and set $T_{\text{hor}}\text{Tw}(M)$ as orthogonal complement to $T_{\text{vert}}\text{Tw}(M) = \ker D\pi \subset T\text{Tw}(M)$.**

Almost complex structure on twistor spaces

REMARK: Let $s \in \Lambda^+(V)$ be a unit vector. Consider the corresponding element $u(s)$ of $\mathfrak{so}(V)$. **Then $u(s)^2 = -1$.** Indeed, $u(s)$ corresponds to a unit imaginary quaterion, and they satisfy $u^2 = -1$.

DEFINITION: We define **the almost complex structure on $\text{Tw}(M)$** as follows. At any point $v \in S\Lambda^+(M)$, **the vector v acts on $T_{\text{hor}}\text{Tw}(M) = \pi^*TM|_v$ as an almost complex structure;** this gives $T_{\text{hor}}\text{Tw}(M)$ a structure of a complex vector bundle. The bundle $T_{\text{vert}}\text{Tw}(M)$ is an oriented, rank 2 bundle with a scalar, such a bundle **has a canonical almost complex structure defined by the counterclockwise rotation.**

REMARK: This almost complex structure **is integrable if and only if the Weyl curvature of M is anti-selfdual.** I am not going to use or explain this observation.

Spin-structures on 4-manifolds and twistor spaces

DEFINITION: A **square root** of a complex line bundle L is a line bundle L_1 such that $L_1^{\otimes 2}$ is isomorphic to L .

Geometric definition: A **spin structure** on an oriented Riemannian 4-manifold M is a square root $\mathcal{V}^{1/2}$ of the vertical tangent bundle $\mathcal{V} = T_{\text{vert}} \text{Tw}(M)$, considered as a complex line bundle.

Standard definition: A **spin structure** on an oriented Riemannian 4-manifold M is a double cover of the principal frame bundle $P_{SO(4)}TM$ with the fibers $\text{Spin}(4) = U(1, \mathbb{H}) \times U(1, \mathbb{H})$.

Spin-structures on 4-manifolds and twistor spaces (2)

THEOREM: These definitions are equivalent.

Proof. Step 1: Twistor space is the quotient of $P_{SO(n)}(TM)$ by the action of $U(2) \subset SO(4)$. Indeed, the twistor space is the space of all oriented orthogonal almost complex structures in TM . **The bundle of unit vectors in the vertical tangent space of $\text{Tw}(M)$ is $P_{SO(4)}TM/U(1, \mathbb{H})$.**

Step 2: The $SO(3)$ -bundle $F := P_{SO(4)}TM/U(1, \mathbb{H})$ is understood as a circle bundle over $\text{Tw}(M) = F/SO(2)$. This bundle is identified with the bundle of unit vectors in the vertical tangent bundle. Therefore, **a square root of \mathcal{V} is the same as the double cover of $P_{SO(4)}TM$ which is non-trivial on fibers.**

Step 3: The quotient map $P_{SO(4)}TM \rightarrow F$ induces an isomorphism of fundamental groups, because its fibers are 3-spheres, which satisfy $\pi_1(S^3) = \pi_2(S^3) = 0$. Therefore, the double covers of F are in bijective correspondence with double covers of $P_{SO(4)}TM$. ■

Spin^c-structures and twistor spaces

DEFINITION: Twistor spheres are fibers of the twistor projection $\pi : \text{Tw}(M) \rightarrow M$.

Geometric definition: A Spin^c-structure on an oriented Riemannian 4-manifold is a complex line bundle L on $\text{Tw}(M)$ such that $c_1(L) = 1$ on each twistor sphere.

Standard definition: A Spin^c-structure on an oriented Riemannian 4-manifold is a circle bundle $\widehat{P}_{SO(n)}(TM) \rightarrow P_{SO(n)}(TM)$ of the orthonormal frame bundle over TM which is compatibly endowed with a structure of a principle Spin^c(4)-bundle, where $\text{Spin}^c(4) := \frac{U(1, \mathbb{H}) \times U(1, \mathbb{H}) \times S^1}{(-1, -1, -1)}$.

Spin^c-structures and twistor spaces (2)

THEOREM: These definitions are equivalent.

Proof. Step 1: Twistor space is the quotient of $P_{SO(n)}(TM)$ by the action of $U(2) \subset SO(4)$. Indeed, the twistor space is the space of all oriented orthogonal almost complex structures in TM .

Step 2: Consider a principal Spin^c(4)-bundle P on M which is also a circle bundle over $P_{SO(n)}(TM)$. Let F be the 2-dimensional fundamental representation of $U(2)$. Consider it as a representation of Spin^c(4) using the natural homomorphism $\frac{U(1, \mathbb{H}) \times U(1, \mathbb{H}) \times S^1}{(-1, -1, -1)} \rightarrow \frac{U(1, \mathbb{H}) \times S^1}{(-1, -1)} = S(2)$. Construct a rank 2 complex vector bundle B on M by taking the associated bundle of this representation. Then $\mathbb{P}B$ is the associated bundle of the quotient of Spin^c(4) by a stabilizer of a line in F , identified with the image of $U(1, \mathbb{H}) \times S^1$ in Spin^c(4), also equal to $U(2)$. **This implies $\text{Tw}(M) = \mathbb{P}B$.** The line bundle $O(1)$, obtained from the fibers of the projection $B \rightarrow \mathbb{P}B$, has degree 1 on twistor spheres. Therefore, **the “standard definition” implies “geometric”.**

Spin^c-structures and twistor spaces (3)

Step 3: Consider the natural projection $P_{SO(n)}(TM) \rightarrow \text{Tw}(M)$ (Step 1). Pulling back the circle bundle of unit vectors in L to $P_{SO(n)}(TM)$, we obtain a circle bundle over $P_{SO(n)}(TM)$. **It remains to construct a free and transitive Spin^c(4)-action on its total space.**

Step 4: The space of fiberwise holomorphic sections of this bundle on the fibers of $\pi : \text{Tw}(M) \rightarrow M$ is a rank 2 complex vector bundle, denoted by B , with $\text{Tw}(M) = \mathbb{P}B$ as in Step 2. By construction, B is a complex spin-representation of $\mathfrak{so}(4)$: the center of the quaternion group acts on fibers of B non-trivially. **Therefore, the pullback of the corresponding circle bundle to $P_{SO(n)}(TM)$ defines a non-trivial bundle on $SO(4)$,** that is, the one representing a non-zero class in $H^1(SO(4), S^1)$.

Step 5: From the exponential exact sequence we obtain an isomorphism $H^1(SO(4), S^1) = H^2(SO(4), \mathbb{Z}) = \mathbb{Z}/2$. Therefore, **the degree 1 line bundle on $\mathbb{C}P^1 = SO(4)/U(2)$ is pulled back to a non-trivial circle bundle on $SO(4)$,** and its total space is naturally isomorphic to Spin^c(4). ■

The Euler class

DEFINITION: Let E be a rank k oriented vector bundle on a manifold M . Consider a section v of E which is transversal to its zero section, that is, the zero set $\zeta(v)$ of v is a smooth codimension k submanifold in M . The **Euler class** $e(E)$ is a cohomology class which is Poincaré dual to $[\zeta(v)] \in H_{n-k}(M, \mathbb{Z})$, where $n = \dim M$. In other words, $e(E)$ is defined by taking a k -cycle, intersecting it with $\zeta(v)$, and counting the intersection points with multiplicities.

REMARK: When k is odd, the multiplication of v by -1 induces the orientation reversal in Z . Therefore, **the Euler class of a bundle of odd rank is a 2-torsion.**

REMARK: For a rank 3 bundle E on a 4-manifold M , its Euler class belongs to $H^3(M, \mathbb{Z})$. Clearly, when E has a non-vanishing section, we have $e(E) = 0$. However, **the converse is not true:** a non-vanishing section of $\Lambda^+(M)$ is the same as an almost complex structure on M , but there are manifolds, such as S^4 , not admitting almost complex structures.

REMARK: Further on, **we will prove that $e(\Lambda^+(M)) = 0$ for any compact oriented 4-manifold M .**

The Gysin exact sequence

Let E be a rank 3 real vector bundle on a compact oriented 4-manifold. Consider **the Gysin exact sequence**

$$\dots \longrightarrow H^{k-3}(M) \xrightarrow{\cup e(E)} H^k(M) \xrightarrow{\pi^*} H^k(Z) \xrightarrow{\pi_*} H^{k-2}(M) \xrightarrow{\cup e(E)} H^{k+1}(M) \longrightarrow \dots$$

(see the third set of handouts to this course). **It can be obtained from the long exact sequence of the pair $(E, E \setminus M)$** , where E denotes the total space of E , and $M \hookrightarrow E$ the zero section:

$$\dots \longrightarrow H^k(E, E \setminus M) \longrightarrow H^k(E) \longrightarrow H^k(E \setminus M) \longrightarrow H^{k+1}(E, E \setminus M) \longrightarrow \dots$$

Indeed, the zero section $M \hookrightarrow E$ is a deformational retract of E , and Z is a deformational retract of $E \setminus M$. The Thom isomorphism $H^{k-3}(M) \longrightarrow H^k(E, E \setminus M)$ makes the first sequence from the second. **The Thom isomorphism is obtained by identifying $H^k(E, E \setminus M)$ with the cohomology with compact support $H_c^k(E)$** , and using the Poincaré duality $H_c^k(E) = (H^{n+3-k}(E))^* = (H^{n+3-k}(M))^* = H^{k-3}(M)$, where $n = \dim M$.

Oriented rank 3 vector bundles

Proposition 1: (C. LeBrun)

Let M be an oriented compact 4-manifold. E a real oriented rank 3 vector bundle, $\pi : Z \rightarrow M$ the S^2 -bundle of unit vectors in E , and $s \in H_2(Z, \mathbb{Z})$ the homology class of the fiber of π . **Then the following are equivalent:**

- (i) the Euler class $e(E) \in H^3(M, \mathbb{Z})$ vanishes
- (ii) there exists a cohomology class $a \in H^2(Z, \mathbb{Z})$ such that $\langle a, s \rangle = 1$.
- (iii) $H_2(Z, \mathbb{Z}) = H_2(M, \mathbb{Z}) \oplus \mathbb{Z}$
- (iv) $|T_2(Z)| = |T_2(M)|$, where $T_2(\cdot)$ denotes the torsion subgroup in $H_2(\cdot, \mathbb{Z})$.

Proof. Step 1: The exactness of the Gysin exact sequence

$$\longrightarrow H^2(Z, \mathbb{Z}) \xrightarrow{\pi_*} H^0(M, \mathbb{Z}) \xrightarrow{\cup e(E)} H^3(M, \mathbb{Z})$$

implies that (i) \Rightarrow (ii). Indeed, **if $e(E) = 0$, then $H^2(Z, \mathbb{Z}) \xrightarrow{\pi_*} H^0(M, \mathbb{Z}) = \mathbb{Z}$ is surjective**, and we chose $a \in H^2(Z, \mathbb{Z})$ such that $\pi_*(a) = 1$. Since π_* is the fiberwise integral, this implies that $\langle a, s \rangle = 1$.

Oriented rank 3 vector bundles (2)

Step 2: Since $H^4(M, \mathbb{Z})$ is free, and $e(E)$ is torsion, the last segment of the Gysin sequence breaks off into a short exact sequence

$$0 \longrightarrow H^4(M, \mathbb{Z}) \xrightarrow{\pi^*} H^4(Z, \mathbb{Z}) \xrightarrow{\pi_*} H^2(M, \mathbb{Z}) \longrightarrow 0,$$

where the image of $H^4(M, \mathbb{Z})$ is generated by the Poincaré dual to the fiber class s . Applying Poincaré duality, we obtain an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot F} H_2(Z, \mathbb{Z}) \longrightarrow H_2(M, \mathbb{Z}) \longrightarrow 0. \quad (*)$$

If (ii) holds, there exists $a \in H^2(Z, \mathbb{Z})$ satisfying $\langle a, s \rangle = 0$; pairing with a provides a left inverse to $\mathbb{Z} \xrightarrow{\cdot F} H_2(Z, \mathbb{Z})$. **Then (ii) implies that (*) splits,** hence gives an isomorphism $H_2(Z, \mathbb{Z}) = H_2(M, \mathbb{Z}) \oplus \mathbb{Z}$. **Therefore, (ii) \Rightarrow (iii).**

Step 3: If $H_2(Z, \mathbb{Z}) = H_2(M, \mathbb{Z}) \oplus \mathbb{Z}$, one also has $T_2(Z) = T_2(M)$, hence (iii) \Rightarrow (iv). On the other hand, the universal coefficients formula implies that the torsion part of cohomology satisfies $T^3(Z) \cong T_2(Z)$ and $T^3(M) \cong T_2(M)$. Therefore, $|T_2(Z)| = |T_2(M)|$ implies that $|T^3(Z)| = |T^3(M)|$. Gysin gives

$$\dots \longrightarrow H^0(M, \mathbb{Z}) \xrightarrow{\cup e(E)} H^3(M, \mathbb{Z}) \longrightarrow H^3(Z, \mathbb{Z}) \longrightarrow H^1(M, \mathbb{Z}) \longrightarrow \dots \quad (**)$$

where $e(E)$ is 2-torsion. Since $H^1(M, \mathbb{Z}) = \text{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z})$ is torsion-free, this means that $T^3(Z) = T^3(M)$ if $e(E) = 0$ and $T^3(Z) = \frac{T^3(M)}{\mathbb{Z}/2}$ otherwise. Therefore, (iv) \Rightarrow (i). ■

Existence of Spin^c -structures

REMARK: Existence of a Spin^c -structure is equivalent to existence of a cohomology class $a \in H^2(M, \mathbb{Z})$ such that $\langle s, a \rangle = 1$ (“geometric definition” of Spin^c -structures). By Proposition 1, **to show that any smooth compact oriented 4-manifold admits a Spin^c -structure it suffices to prove that $e(\Lambda^+(M)) = 0$.**

The following theorem is originally proven by Hirzebruch-Hopf. The proof I give is due to C. LeBrun.

THEOREM: **Any smooth compact oriented 4-manifold admits a Spin^c -structure.**

Existence of Spin^c -structures (2)

THEOREM: Any smooth compact oriented 4-manifold admits a Spin^c -structure.

Proof. Step 1: It would suffice to show that any of the equivalent conditions (i)-(iv) of Proposition 1 hold. We will prove that $|T_2(Z)| = |T_2(M)|$. From (***) it is apparent that $|T_2(Z)| \leq |T_2(M)|$. Therefore, **the equality $|T_2(Z)| = |T_2(M)|$ would follow if we prove that the homomorphism $\pi_* : T_2(Z) \rightarrow T_2(M)$ is surjective.**

Step 2: Using the exponential exact sequence, we can construct a complex line bundle L with any given class $c_1(L) \in H^2(M, \mathbb{Z})$. Since $c_2(L)$ is Poincaré dual to the zero set of its general section, which is smooth, **any element in $H_2(M, \mathbb{Z})$ can be represented by a smooth oriented 2-manifold $S \subset M$.**

Step 3: Let $\Sigma \subset M$ be a smooth compact oriented 2-manifold representing a torsion class $[\Sigma] \in T_2(M) \subset H_2(M, \mathbb{Z})$. Since the homological self-intersection of Σ vanishes, its normal bundle $N\Sigma$ has trivial Euler class. **Since $N\Sigma$ is orientable, it is also trivial.**

Step 4: Define a complex structure on $TM|_{\Sigma}$ by taking a counterclockwise rotation on $T\Sigma$ and $N\Sigma$ identified with $T\Sigma^{\perp} \subset TM|_{\Sigma}$. This allows us to lift Σ to $\text{Tw}(M)$ obtaining an embedded 2-submanifold $\hat{\Sigma} \subset \text{Tw}(M)$ projecting to Σ . Since the normal bundle $N\Sigma$ is trivial, **we have** $\langle c_1(T_{\text{hor}} \text{Tw}(M)), \hat{\Sigma} \rangle = \langle c_1(N\Sigma \oplus T\Sigma), \Sigma \rangle = \chi(\Sigma) = 2(1 - g)$.

Step 5: Consider the homology class $A := [\hat{\Sigma}] + (g - 1)s \in H_2(\text{Tw}(M), \mathbb{Z})$, where s denotes the homology class of the twistor sphere. Clearly, $\pi_*(A) = \Sigma$. **We are going to show that A is a torsion class.** This would follow if we prove that $\langle a, A \rangle = 0$ for any $a \in H^2(\text{Tw}, \mathbb{R})$.

Step 6: Let $W \in H^2(Z, \mathbb{R})$ be any cohomology class which is non-trivial on the twistor sphere, such as $c_1(T_{\text{vert}}Z)$. The Gysin exact sequence

$$0 \rightarrow H^2(M) \xrightarrow{\pi^*} H^2(Z) \xrightarrow{\pi_*} H^0(M) \xrightarrow{\cup e(E)} H^3(M)$$

yields $H^2(Z, \mathbb{R}) = \pi^*(H^2(M, \mathbb{R})) \oplus \mathbb{R}W$, because $e(E)$ is torsion. Pairing A with elements from $\pi^*(H^2(M, \mathbb{R}))$ vanishes because $\pi_*A = [\Sigma]$ is torsion. Since $T_{\text{hor}} \text{Tw}(M)$ restricted to the twistor spheres is $\mathcal{O}(1) \oplus \mathcal{O}(1)$, we can take $W = c_1(T_{\text{hor}} \text{Tw}(M))$; by construction, $\langle A, W \rangle = 0$. **Therefore, A is torsion, and the natural map $\pi_*T_2(Z) \rightarrow T_2(M)$ is surjective.** By Step 1, M admits a Spin^c -structure. ■