Seiberg-Witten invariants,

lecture 5: the Dirac operator

IMPA, sala 236

Misha Verbitsky, February 9, 2026, 17:00

`http://verbit.ru/IMPA/SW-2026/`

Ehresmann connections

DEFINITION: Let $\pi : M \longrightarrow Z$ be a smooth submersion, with $T_\pi M$ **the bundle of vertical tangent vectors** (vectors tangent to the fibers of π). An **Ehresmann connection** on π is a sub-bundle $T_{hor} M \subset TM$ such that $TM = T_{hor} M \oplus T_\pi M$. The **parallel transport** along the path $\gamma : [0, a] \longrightarrow Z$ associated with the Ehresmann connection is a diffeomorphism

$$V_t : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(t))$$

smoothly depending on $t \in [0, a]$ and satisfying $\frac{dV_t}{dt} \in T_{hor} M$.

CLAIM: Let $\pi : M \longrightarrow Z$ be a smooth fibration with compact fibers. Then **the parallel transport, associated with the Ehresmann connection, always exists.**

Proof: Follows from existence and uniqueness of solutions of ODEs. ■

Principal bundles (reminder)

DEFINITION: Let G be a Lie group. **Principal G-bundle** over a manifold M is a smooth fibration $P \mapsto M$ with a smooth G-action which acts freely and transitively on fibers.

EXAMPLE: Frame bundle on a smooth n-manifold M is the bundle of all frames (basises) in $T_x M$, for all $x \in M$.

DEFINITION: Let $H \longrightarrow G$ be a group homomorphism, and P a principal H-bundle. Then the quotient $P_G := P \times G/H$ (with H acting on both components in a natural way) is called **an associated principal bundle**, and P is called **reduction of the principal G-bundle P_G to the group H**.

DEFINITION: Let G be a Lie group, and $G \longrightarrow GL(n, \mathbb{R})$ a group homomorphism. **A G-structure on a manifold M** is a reduction of the principal frame bundle to G.

DEFINITION: Let G be a Lie group, V its representation, and P a principal G-bundle on M. The quotient $P \times V/G$ is a vector bundle over M, called **the associated vector bundle**.

Connections

DEFINITION: Let B be a vector bundle on M. **Connection** on B is a differential operator $\nabla: B \to B \otimes \Lambda^1 M$ such that $\nabla(fb) = f\nabla(b) + b \otimes df$, for any $f \in C^\infty M$ and any section b of B.

DEFINITION: Let $\pi: P \to M$ be a principal G-bundle. **A connection** on P is a G-invariant Ehresmann connection.

REMARK: We are going to construct **an equivalence between the connections in the principal frame bundles and connections in the corresponding vector bundles.**

REMARK: Let X be a manifold with smooth action of G, and $P_X := P \times_G X \stackrel{\pi_X}{\to} M$ the associated fibration. Consider the exact sequence

$$0 \to T_{\pi_X} P_X \to TP_X \to \pi_X^* TM \to 0. \quad (*)$$

A G-invariant splitting of the exact sequence $0 \to T_\pi P \to TP \to \pi^* TM \to 0$ gives a splitting of (*), because $T_{(x,p)} P_X = \frac{T_p P \oplus T_x X}{\mathfrak{g}}$, where $\mathfrak{g}$ is the Lie algebra of G. Therefore, **A connection in a principal bundle defines an Ehresmann connection in every associated fibration**.

The horizontal lift

DEFINITION: Let $\pi : P \longrightarrow M$ be a smooth submersion. **A lift** of a vector field $X \in TM$ is a vector field $\tilde X$ on P such that for all $x \in P$ the differential $d\pi$ takes $\tilde X|_x \in T_xP$ to $X\Big|_{\pi(x)} \in T_{\pi(x)}M$.

DEFINITION: Let $TP = T_{\text{hor}}P \oplus T_\pi P$ be an Ehresmann connection. Since the differential $D\pi : TP \to TM$ defines a bijective map $T_{\text{hor}}P \to TM$, for every vector field $X \in TM$ there exists a unique lift $X_{\text{hor}} \in T_{\text{hor}}P$. It is called **the horizontal lift** of X.

CLAIM: Let $X \in TM$ be a vector field, and γ its integral curve. **Then the parallel transport along the Ehresmann connection associated with γ is equal to the corresponding diffeomorphism flow e^X.**

Proof: Clear. ■

Ehresmann connection on a product

REMARK: Let $\pi_1 : M_1 \longrightarrow Z$, $\pi_2 : M_2 \longrightarrow Z$ be smooth submersions with fibers F_1, F_2. Clearly, **the fibered product $\pi : M_1 \times_Z M_2 \longrightarrow Z$ a smooth submersion with fiber $F_1 \times F_2$.**

PROPOSITION: Consider standard exact sequence, $0 \longrightarrow T_\pi(M_1 \times_Z M_2) \longrightarrow T(M_1 \times_Z M_2) \longrightarrow \pi^* TZ \longrightarrow 0$. Let $\Pi_1 : M_1 \times_Z M_2 \longrightarrow M_1$, $\Pi_2 : M_1 \times_Z M_2 \longrightarrow M_2$ be the projection maps. Define the bundle

$$B := \{v \in T_{m_1,m_2}(M_1 \times_Z M_2) \ \mid\ D\Pi_1(v) \in T_{\mathrm{hor}} M_1, \text{ and } D\Pi_2(v) \in T_{\mathrm{hor}} M_2\}.$$

Then

(i) **B is isomorphic to $\pi^* TZ$ and defines a splitting of the exact sequence (**),** that is, **an Ehresmann connection on the fibration $M_1 \times_Z M_2 \to Z$.**

(ii) Conversely, consider an Ehresmann connection $T(M_1 \times_Z M_2) = T_\pi(M_1 \times_Z M_2) \oplus T_{\mathrm{hor}}(M_1 \times_Z M_2)$. Then the image of the differential $D\Pi_i$ in TM_i satisfies $D\Pi_i(T_{\mathrm{hor}}(M_1 \times_Z M_2)) \oplus T_{\pi_i} M_i = TM_i$, that is, **defines an Ehresmann connection on the fibration $\pi_i M_i \longrightarrow Z$.**

Proof: Left as an exercise. ■

REMARK: This defines **a bijective correspondence between the Ehresmann connections on the fibration $\pi : M_1 \times_Z M_2 \longrightarrow Z$ and the pairs of Ehresmann connections on $\pi_1 : M_1 \longrightarrow Z$, $\pi_2 : M_2 \longrightarrow Z$.**

Linear connections

DEFINITION: Let B be a vector bundle on M and $\pi: \operatorname{Tot} B \longrightarrow M$ its total space. An Ehresmann connection $T \operatorname{Tot} B = T_{\text{hor}} \operatorname{Tot} B \oplus T_\pi \operatorname{Tot} B$ on π is called **homogeneous** if the decomposition $T \operatorname{Tot} B = T_{\text{hor}} \operatorname{Tot} B \oplus T_\pi \operatorname{Tot} B$ is preserved by the differential of homothety map $v \longrightarrow \lambda v$.

DEFINITION: Consider the connection on $\operatorname{Tot} B \times_M \operatorname{Tot} B \longrightarrow \operatorname{Tot} B$ induced by the Ehresmann connection on $\operatorname{Tot} B$. We say that an Ehresmann connection $T \operatorname{Tot} B = T_{\text{hor}} \operatorname{Tot} B \oplus T_\pi \operatorname{Tot} B$ is **additive** if the differential of the addition map $\operatorname{Tot} B \times_M \operatorname{Tot} B \longrightarrow \operatorname{Tot} B$ takes horizontal vectors to horizontal vectors. We say that an Ehresmann connection is **linear** if it is additive and homogeneous.

PROPOSITION: Let B be a vector bundle on M, $\pi: \operatorname{Tot} B \longrightarrow M$ its total space, and $T \operatorname{Tot} B = T_\pi \operatorname{Tot} B \oplus T_{\text{hor}} \operatorname{Tot} B$ an Ehresmann connection on $\operatorname{Tot} B$. Then this connection **is linear if and only if the parallel transport map $V_t: \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(t))$ taking a fiber of B to another fiber of B is linear** with respect to the structure of the vector space on these fibers.

Proof: A connection is homogeneous if and only if the the horizontal lift of a vector field commutes with homothety, and additive if and only if it compatible with the group structure. ■

Dual connections

DEFINITION: Let $\pi : P \to M$ be a fibration equipped with an Ehresmann connection. Then the horizontal lifts of vector fields $X \in TM$ define diffeomorphisms of P preserving the fibers. These diffeomorphisms are called **parallel transports along the connection**.

CLAIM: Let $T \operatorname{Tot} B = T_{\mathrm{hor}} \operatorname{Tot} B \oplus T_\pi \operatorname{Tot} B$ be a linear Ehresmann connection, and $\operatorname{Tot} B^* \times_M \operatorname{Tot} B \xrightarrow{\kappa} \operatorname{Tot} C^\infty M = \mathbb{C} \times M$ the natural pairing. Then **there exists a unique Ehresmann connection on $\operatorname{Tot} B^*$ such that the map κ takes horizontal vectors to horizontal vectors.**

Proof: The parallel transport map in $\operatorname{Tot} B^*$ is dual to the parallel transport in $\operatorname{Tot} B$. ■

Fiberwise linear functions

DEFINITION: Let B be a vector bundle, and f a function on $\operatorname{Tot} B$ which is linear on all fibers of π. Such a function is called **fiberwise linear**.

CLAIM: Let B be a vector bundle over M. An Ehresmann connection on $\operatorname{Tot} B$ **is linear if and only if the Lie derivative of a fiberwise linear function along a horizontal vector field is again fiberwise linear.**
Proof: The diffeomorphism flow associated with a horizontal lift induces a linear map on fibers if and only if the Ehresmann connection is linear. ■

REMARK: Clearly, **every section of B defines a fiberwise linear function on $\operatorname{Tot} B^*$,** and, conversely, **every fiberwise linear function on $\operatorname{Tot} B^*$ is associated with a smooth section of B.**

Linear vector fields

PROPOSITION: Let $\operatorname{Tot} B$ be a total space of a vector bundle equipped with a linear Ehresmann connection This also defines a linear Ehresmann connection on $\operatorname{Tot} B^*$. Given a vector field $X \in TM$, denote by $X_{\text{hor}} \in T \operatorname{Tot} B^*$ its horizontal lift. Consider a section b of B as a fiberwise linear function on $\operatorname{Tot} B^*$. The derivative $\operatorname{Lie}_{X_{\text{hor}}} b$ is fiberwise linear on $\operatorname{Tot} B^*$, hence it can be interpreted as a section of B. **Then the map $\nabla_X b := \operatorname{Lie}_{X_{\text{hor}}} b$ defines a connection on the vector bundle B.**

Proof: Follows from the Leibnitz formula for the Lie derivative. ■

PROPOSITION: All connections on the vector bundle B are obtained this way.
Proof: Left as an exercise. ■

Connections in frame bundles

REMARK: Let $P_{GL(n,\mathbb{R})}TM$ be a frame bundle on TM, considered as a principal bundle. We consider $P_{GL(n,\mathbb{R})}TM$ as an open subset in $(TM)^n$, fibered over M with the fiber $(\mathbb{R}^n)^n$. Then, **any Ehresmann connection in TM gives an Ehresmann connection in $P_{GL(n,\mathbb{R})}TM \subset TM$.** By construction, **this connection is $GL(n,\mathbb{R})$-invariant.**

The following proposition gives an inverse correspondence.

PROPOSITION: Let V be a representation of G, and $\pi: P \to M$ a principal G-bundle equipped with an Ehresmann connection. Consider the natural Ehresmann connection on the associated vector bundle $P \times_G V$ defined earlier today. **Then this connection is linear**.

Proof: The action of G on V is compatible with the homothety and with the parallel transport, hence the parallel transport in the vector bundle $P \times_G V$ preserves the linear structure. ■

Spin^c-group (reminder)

DEFINITION: Let V be a real vector space with a scalar product, and let $z \in \text{Spin}(V)$ be the non-trivial element in the kernel of the homomorphism $\text{Spin}(V) \to SO(V)$. Then $\text{Spin}^c(V) := \frac{\text{Spin}(V)\times U(1)}{(z,-1)}$ is called **the Spin^c-group.**

REMARK: Let $S(V)$ be the Clifford module, that is, the spinorial representation associated with V. Consider the natural map $\text{Cl}(V) \to U(S(V) \otimes \mathbb{C})$. Then $\text{Spin}^c(V)$ is the image of $\text{Spin}(V) \times U(1)$ in $U(S(V) \otimes \mathbb{C})$. Therefore, **the space $U(S(V) \otimes \mathbb{C})$ admits a faithful action of the group $\text{Spin}^c(V)$.**

DEFINITION: A Spin^c-structure on a bundle B, $\text{rk}(B) = n$, is a reduction of its structure group to $\text{Spin}^c(n)$.

Spin^c-structures (reminder)

REMARK 1: From the exponential exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0$, **we obtain an isomorphism $H^1(M, S^1) = H^2(M, \mathbb{Z})$.** Let c_1 denote the map $H^1(M, S^1) \to H^2(M, \mathbb{Z})$.

THEOREM: Let M be a manifold, and $R: H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}/2)$ be the reduction mod 2 map. An oriented vector bundle B admits a Spin^c-structure **if and only if $w_2(B) \in \operatorname{im} R$.**

Proof. Step 1: Clearly, Spin$^c(V)$ is a double cover of $SO(V)$. Consider the exact sequence

$$H^1(M, \mathbb{Z}/2) \to H^1(M, \mathrm{Spin}^c(n)) \to H^1(M, SO(n)) \oplus H^1(M, S^1) \xrightarrow{w_2 + c_1} H^2(M, \mathbb{Z}/2).$$

It follows that **a principal Spin$^c(n)$-bundle is a principal $SO(n) \times S^1$-bundle P such that $w_2 + c_1(P) = 0$.**

Step 2: In other words, a principal $SO(n) \times S^1$-bundle is a pair of a principal $SO(n)$-bundle B and a principal S^1-bundle L, such that $w_2(B) + c_1(L) = 0$ mod 2.

Step 3: From Remark 1, we obtain that a principal S^1-bundle is uniquely determined by $c_1(L) \in H^2(M, \mathbb{Z})$. Therefore, a vector bundle B admits a Spin^c-structure **if and only if there exists $c_1(L) \in H^2(M, \mathbb{Z})$ such that $w_2(B) + c_1(L) = 0 \mod 2$.** ■

Levi-Civita connection on the spinor bundle

DEFINITION: A bundle of spinors on a spin-manifold M is a vector bundle associated to the principal Spin(n)-bundle and a spin representation.

REMARK: Let $Z \subset G$ be a discrete central subgroup. Clearly, an G-invariant Ehresmann connection on G-bundle P is the same as a G/Z-invariant connection on P/Z, considered as a principal P/Z-bundle. Therefore, **the Levi-Civita connection on $P_{SO(n)}TM$ is naturally extended to its double cover Spin(n)-bundle**.

DEFINITION: As shown in the first part of this lecture, **this defines a connection in any of the associated vector bundles.** The corresponding connection on the spinor bundle is also called **the Levi-Civita connection**.

The Dirac operator

DEFINITION: Consider the map $TM \otimes \text{Spin} \longrightarrow \text{Spin}$ induced by the Clifford multiplication. One defines **the Dirac operator** $D : \text{Spin} \longrightarrow \text{Spin}$ as a composition of $\nabla : \text{Spin} \longrightarrow \Lambda^1 M \otimes \text{Spin} = TM \otimes \text{Spin}$ and the multiplication.

DEFINITION: A **harmonic spinor** is a spinor ψ such that $D(\psi) = 0$.

THEOREM: (Bochner's vanishing) A harmonic spinor ψ on a compact manifold with vanishing scalar curvature $Sc = Tr(\text{Ric})$ **satisfies $\nabla\psi = 0$.**

Proof: (More details will be given in the end of this lecture.) The **coarse Laplacian** $\nabla^*\nabla$ is expressed through the Dirac operator using the **Lichnerowitz formula** $\nabla^*\nabla - D^2 = -\frac{1}{4}Sc$. When these two operators are equal, **any harmonic spinor ψ lies in $\ker \nabla^*\nabla$, giving $(\psi, \nabla^*\nabla\psi) = (\nabla\psi, \nabla\psi) = 0$.**

■

Spin^c-structures: an associated bundle

DEFINITION: Let V be a vector space, $S(V)$ a spin-representation of Spin(V), and $S_{\mathbb{C}}V := S(V)\otimes_{\mathbb{R}}\mathbb{C}$ its complexification, considered as a Spin$^c(V)$-representation. **A complex spinor bundle** associated with a principal Spin$^c(V)$-bundle P is $P \times_{\text{Spin}^c(V)} S_{\mathbb{C}}V$.

REMARK: Consider a Spin$^c(TM)$-bundle on M. The corresponding group is Spin$^c(n) = \frac{\text{Spin}(n)\times U(1)}{\pm 1}$ To define a connection on Spin$^c(TM)$-bundle, **we need to define it on its double cover Spin$(n) \times U(1)$.**

DEFINITION: Let L denote the line bundle associated with the $U(1)$-bundle associated with the Spin^c-structure. It is called **the determinant bundle** of the Spin^c-structure. Clearly, **L is a complex hermitian line bundle.**

DEFINITION: **A spinorial connection** on the Spin$^c(TM)$-bundle is obtained from the Levi-Civita connection on its Spin(n)-component and a unitary connection on the unitary complex line bundle L.

Gaussian curvature

CLAIM: Let ∇ be a Levi-Civita connection on a Riemannian manifold, and $R \in T^*M^{\otimes 3} \otimes TM$ its curvature tensor. Using an isomorphism $TM \cong T^*M$ given by the metric, we may consider R as an element in $T^*M^{\otimes 4}$. **Then R is a section of $\operatorname{Sym}^2(\Lambda^2 T^*M)$, antisymmetric in 1,2 and 3,4 indices.**

DEFINITION: Let V be a vector space with non-degenerate scalar product g. **A trace** $\operatorname{Tr}_{12}: V^{\otimes^n} \longrightarrow V^{\otimes^{n-2}}$ is defined as a map dual to the multiplication $A \longrightarrow g \otimes A$. **The trace in i-th and j-th indices**, denoted as $\operatorname{Tr}_{ij}: V^{\otimes^n} \longrightarrow V^{\otimes^{n-2}}$, is defined as a map which acts in the i-th and j-th component as Tr_{12} on the first two.

DEFINITION: Gaussian curvature of a Riemannian manifold is a scalar $\operatorname{Tr}_{13}\operatorname{Tr}_{24}(R)$, where R is the Riemannian curvature.

Clifford multiplication in $\mathrm{Sym}^2(\Lambda^2 V)$

There are lots of constants missing from now on: 1/2, 1/4 and so on. It is a good exercise to find every place where I omit the constant and put it back.

LEMMA 1: Let $R \in \mathrm{Sym}^2(\Lambda^2 V)$, where V is a space with scalar product g. Denote the Clifford multiplication as $\sigma : V^{\otimes^4} \longrightarrow \mathrm{Cl}(V)$. **Then**

$$\sigma(R) = \mathrm{Tr}_{13}\,\mathrm{Tr}_{24}\,R + \sigma(\mathrm{Alt}(R)),$$

where $\mathrm{Alt}: \mathrm{Sym}^2(\Lambda^2 V) \longrightarrow \Lambda^4 V$ is the exterior product map.

Proof: Let $x, y, z, t \in V$, and $R(x, y, z, t) := (xy - yx)(zt - tz) + (zt - tz)(xy - yx)$ be the corresponding element in $\mathrm{Sym}^2(\Lambda^2 V)$. Then

1. If x, y, z, t are pairwise orthogonal, we have $\sigma(R(x, y, z, t)) = \sigma(\mathrm{Alt}(R))$, because x, y, z, t anticommute in the Clifford algebra.

2. If x, y, z are pairwise orthogonal, and $y = t$, then $xy - yx$ anticommutes with $zt - tz$, hence $\sigma(R(x, y, z, t)) = 0$.

3. If x, y are orthogonal, $y = t$ and $x = z$, we have

$$\sigma(R(x, y, z, t)) = \sigma((xy - yx)^2) = g(x, x)g(y, y).$$

■

Laplacian and rough Laplacian

REMARK: Let $D: S \longrightarrow S$ be the Dirac operator, and $x_i \in TM$ an orthonormal frame. **Then $D(s) = \sum_i \sigma(x_i, \nabla_{x_i}(s))$, where $\sigma: TM \otimes S \longrightarrow S$ is Clifford multiplication.**

COROLLARY: Let $\Theta \in \Lambda^2 M \otimes \operatorname{End}(S)$ be the curvature of S. Then

$$D^2(s) = \sum_{i,j} \sigma(x_i x_j, \nabla_{x_i} \nabla_{x_j} s) = \sum_{i,j} \sigma(x_i x_j, \Theta_{x_i, x_j} s) + \sum_{i,j} \sigma(x_i x_j + x_j x_i, \nabla_{x_i} \nabla_{x_j} s).$$

Since $\sigma(x_i x_j + x_j x_i, v) = g(x_i, x_j) v$, this gives

$$D^2(s) = \sigma(\Theta, s) + \sum_i \nabla_{x_i} \nabla_{x_i} s.$$

DEFINITION: **Rough Laplacian** on a bundle B with connection on a Riemannian manifold is defined as $Đ(s) := \operatorname{Tr}_{12}(\nabla^2 s)$.

REMARK: The previous corollary is then rewritten as $D^2(s) = \sigma(\Theta, s) + Đ(s)$.

REMARK: $\int (Đ(s), s) \operatorname{Vol} = \int (\nabla s, \nabla s) \operatorname{Vol}$. Therefore, **on a compact manifold, $Đ(s) = 0 \Leftrightarrow \nabla(s) = 0$.**

The Weitzenböck formula

THEOREM: (Lichnerowicz-Weitzenböck formula)
Let M be a Riemannian manifold with spin structure, $\text{Đ} : S \longrightarrow S$ the rough Laplacian, Sc multiplication by the scalar curvature, and $D : S \longrightarrow S$ the Dirac operator. **Then $D^2 = \text{Đ} + \mathrm{Sc}$.**

Proof: $D^2(s) = \sigma(\Theta, s) + \text{Đ}(s)$, as shown above, and $\sigma(\Theta, s) = \mathrm{Sc}(s) + \sigma(\mathrm{Alt}(R))$ by Lemma 1. The last term vanishes, because $\mathrm{Alt}(R)$ (Bianchi identity). ■

REMARK: Clearly,

$$g(\text{Đ}(s), s) = \mathrm{Tr}_{12}(\nabla^2(s), s) = g(\nabla(s), \nabla(s)).$$

This gives $\int_M g(\text{Đ}(s), s) = \int_M g(\nabla(s), \nabla(s))$. Therefore, **on a compact manifold $\text{Đ}(s) = 0$ is equivalent to $\nabla(s) = 0$.**

Bochner vanishing for harmonic spinors

COROLLARY: (Bochner vanishing)
Let M be a compact Riemannian manifold with non-negative scalar curvature. **Then $\nabla(s) = 0$ for any harmonic spinor s.** If, in addition, $\mathrm{Sc} > 0$ somewhere on M, then $s = 0$.

Proof: Lichnerowicz-Weitzenböck formula gives

$$0 = g(D^2(s), s) = g(Đ(s), s) + \int_M \mathrm{Sc} \cdot g(s, s) = \int_M g(\nabla(s), \nabla(s)) + \int_M \mathrm{Sc} \cdot g(s, s).$$

The first term vanishes. Moreover, $s = 0$ on the set $U \subset M$ where $\mathrm{Sc} > 0$. Then $s = 0$ because $\nabla(s) = 0$. ■