Seiberg-Witten invariants,

lecture 6: the Seiberg-Witten equations

IMPA, sala 236

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$\mathrm{Cl}(W \oplus W^*) = \mathrm{Mat}(\Lambda^* W)$ **(reminder)**

THEOREM: Let $V := W \oplus W^*$, with the usual pairing $\langle (x+\xi), (x'+\xi')\rangle = \xi(x') + \xi'(x)$. **Then $\mathrm{Cl}(V)$ is naturally isomorphic to $\mathrm{Mat}(\Lambda^* V^*)$.**

Proof. Step 1: Consider the convolution map $W \otimes \Lambda^i W^* \longrightarrow \Lambda^{i-1} W^*$, with $v \otimes \xi \longrightarrow \xi(v, \cdot, \cdot, \ldots, \cdot)$ denoted by $v, \xi \longrightarrow i_v(\xi)$ and the extertior multiplication map $W^* \otimes \Lambda^i W^* \longrightarrow \Lambda^{i-1} W^*$, with $\nu \otimes \xi \longrightarrow \nu \wedge \xi$, denoted by $\nu, \xi \longrightarrow e_\nu(\xi)$. Let $V \otimes \Lambda^* W^* \xrightarrow{\Gamma} \Lambda^* W^*$ map $(v, \nu) \otimes \xi$ to $i_v(\xi) + e_\nu(\xi)$. Clearly, all i_v pairwise anticommute, all e_ν pairwise anticommute, and the anticommutator $\{i_v, e_\nu\}$ is a scalar operator of multiplication by a number $\nu(v)$.

To prove the last assertion without any calculations, we notice that i_v **is an odd derivation of the Grassmann algebra, e_ν is a linear operator, and a commutator of a derivation and a linear operator is linear,** hence one has

$$\{i_v, e_\nu\}(a) = \{i_v, e_\nu\}(1) \wedge a = \nu(v) \cdot a.$$

Step 2: These relation imply that the map $V \otimes \Lambda^* W^* \xrightarrow{\Gamma} \Lambda^* W^*$, is extended to a homomorphism $\mathrm{Cl}(V) \longrightarrow \mathrm{Mat}(\Lambda^* W^*)$. I's not hard to show that this map is surjective. **Since $\dim \mathrm{Cl}(V) = 2^{\dim V} = 2^{2\dim W} = \dim \mathrm{Mat}(\Lambda^* W^*)$, this also implies that $\mathrm{Cl}(V) \cong \mathrm{Mat}(\Lambda^* W^*)$.** ∎

Spin^c and the complex Clifford algebra

Consider the homomorphism $U(n) \to SO(2n) \times U(1)$ taking a matrix $A \in U(n)$ to its image under the natural embedding $U(n) \subset SU(2n)$ times its determinant in $\mathbb{C}$.

CLAIM: This defines a homomorphism $U(n) \times SO(2n) \times U(1)$; lifting it to a $\mathbb{Z}/2$-covering Spin$^c(2n) \to SO(2n) \times U(1)$ **gives an embedding $U(n) \hookrightarrow$ Spin$^c(2n)$.**

Proof: We have just shown that Cl$(n,n) =$ Mat$(2^n, \mathbb{R})$. Let $V = (\mathbb{R}^{2n}, I, g)$ be a complex Hermitian vector space. Clearly, g restricted to $V^{0,1} \oplus V^{0,1} = V \otimes_{\mathbb{R}} \mathbb{C}$ defines a pairing between $V^{0,1}$ and $V^{0,1}$, hence Cl$(V \otimes_{\mathbb{R}} \mathbb{C}, g)$ is naturally identified with Mat$(2^n, \mathbb{C})$. Automorphisms of V act on Cl$(V \otimes_{\mathbb{R}} \mathbb{C}, g)$ and on its Clifford module $\Lambda^{*,0}(V)$, defining a natural homomorphism $U(n) \to$ Spin$^c(V)$. ■

COROLLARY: Any almost complex manifold **is equipped with a natural Spin^c-structure.** ■

CLAIM: Let (M, I) be an almost complex manifold, and $P \to M$ the Spin$^c(V)$-bundle associated with its almost complex structure as above. **Then its complex spinor bundle is $\bigoplus \Lambda^{*,0} M$.** ■

Gaussian curvature (reminder)

CLAIM: Let ∇ be a Levi-Civita connection on a Riemannian manifold, and $R \in T^*M^{\otimes 3} \otimes TM$ its curvature tensor. Using an isomorphism $TM \cong T^*M$ given by the metric, we may consider R as an element in $T^*M^{\otimes 4}$. **Then R is a section of $\operatorname{Sym}^2(\Lambda^2 T^*M)$, antisymmetric in 1,2 and 3,4 indices.**

DEFINITION: Let V be a vector space with non-degenerate scalar product g. **A trace** $\operatorname{Tr}_{12} : V^{\otimes^n} \longrightarrow V^{\otimes^{n-2}}$ is defined as a map dual to the multiplication $A \longrightarrow g \otimes A$. **The trace in i-th and j-th indices**, denoted as $\operatorname{Tr}_{ij} : V^{\otimes^n} \longrightarrow V^{\otimes^{n-2}}$, is defined as a map which acts in the i-th and j-th component as Tr_{12} on the first two.

DEFINITION: **Gaussian curvature** of a Riemannian manifold is a scalar $\operatorname{Tr}_{13}\operatorname{Tr}_{24}(R)$, where R is the Riemannian curvature.

Clifford multiplication in $\mathrm{Sym}^2(\Lambda^2 V)$ (reminder)

There are lots of constants missing from now on: 1/2, 1/4 and so on. It is a good exercise to find every place where I miss the constant and put it back.

LEMMA 1: Let $R \in \mathrm{Sym}^2(\Lambda^2 V)$, where V is a space with scalar product g. Denote the Clifford multiplication as $\sigma : V^{\otimes^4} \longrightarrow \mathrm{Cl}(V)$. **Then**

$$\sigma(R) = \mathrm{Tr}_{13}\,\mathrm{Tr}_{24}\,R + \sigma(\mathrm{Alt}(R)),$$

where $\mathrm{Alt} : \mathrm{Sym}^2(\Lambda^2 V) \longrightarrow \Lambda^4 V$ is the exterior product map.

Proof: Let $x, y, z, t \in V$, and $R(x, y, z, t) := (xy - yx)(zt - tz) + (zt - tz)(xy - yx)$ be the corresponding element in $\mathrm{Sym}^2(\Lambda^2 V)$. Then

1. If x, y, z, t are pairwise orthogonal, we have $\sigma(R(x, y, z, t)) = \sigma(\mathrm{Alt}(R))$, because x, y, z, t anticommute in the Clifford algebra.

2. If x, y, z are pairwise orthogonal, and $y = t$, then $xy - yx$ anticommutes with $zt - tz$, hence $\sigma(R(x, y, z, t)) = 0$.

3. If x, y are orthogonal, $y = t$ and $x = z$, we have

$$\sigma(R(x, y, z, t)) = \sigma((xy - yx)^2) = g(x, x)g(y, y).$$

■

Laplacian and rough Laplacian (reminder)

REMARK: Let $D: S \longrightarrow S$ be the Dirac operator, and $x_i \in TM$ an orthonormal frame. **Then $D(s) = \sum_i \sigma(x_i, \nabla_{x_i}(s))$, where $\sigma: TM \otimes S \longrightarrow S$ is Clifford multiplication.**

COROLLARY: Let $\Theta \in \Lambda^2 M \otimes \operatorname{End}(S)$ be the curvature of S. Then

$$D^2(s) = \sum_{i,j} \sigma(x_i x_j, \nabla_{x_i}\nabla_{x_j} s) = \sum_{i,j} \sigma(x_i x_j, \Theta_{x_i,x_j} s) + \sum_{i,j} \sigma(x_i x_j + x_j x_i, \nabla_{x_i}\nabla_{x_j} s).$$

Since $\sigma(x_i x_j + x_j x_i, v) = g(x_i, x_j) v$, this gives

$$D^2(s) = \sigma(\Theta, s) + \sum_i \nabla_{x_i}\nabla_{x_i} s.$$

DEFINITION: Rough Laplacian on a bundle B with connection on a Riemannian manifold is defined as $Đ(s) := \operatorname{Tr}_{12}(\nabla^2 s)$.

REMARK: The previous corollary is then rewritten as $D^2(s) = \sigma(\Theta, s) + Đ(s)$.

REMARK: $\int (Đ(s), s) \operatorname{Vol} = \int (\nabla s, \nabla s) \operatorname{Vol}$. Therefore, **on a compact manifold, $Đ(s) = 0 \Leftrightarrow \nabla(s) = 0$.**

The Bochner-Weitzenböck formula (reminder)

THEOREM: (Bochner-Lichnerowicz-Weitzenböck formula)
Let M be a Riemannian manifold with spin structure, Đ : $S \longrightarrow S$ the rough Laplacian, Sc multiplication by the scalar curvature, and $D: S \longrightarrow S$ the Dirac operator. **Then $D^2 =$ Đ $+$ Sc.**

Proof: $D^2(s) = \sigma(\Theta, s) +$ Đ(s), as shown above, and $\sigma(\Theta, s) = \mathrm{Sc}(s) + \sigma(\mathrm{Alt}(R))$ by Lemma 1. The last term vanishes, because $\mathrm{Alt}(R)$ (Bianchi identity). ■

REMARK: Clearly,

$$g(\text{Đ}(s), s) = \mathrm{Tr}_{12}(\nabla^2(s), s) = g(\nabla(s), \nabla(s)).$$

This gives $\int_M g(\text{Đ}(s), s) = \int_M g(\nabla(s), \nabla(s))$. Therefore, **on a compact manifold Đ$(s) = 0$ is equivalent to $\nabla(s) = 0$.**

The Bochner-Weitzenböck formula for Spin^c-manifolds

THEOREM: Let M be a Riemannian manifold with Spin^c-structure, S the bundle of Spin^c-spinors, $Đ: S \longrightarrow S$ the rough Laplacian, ω the curvature of the corresponding complex line bundle, Sc the multiplication by the scalar curvature, and $D: S \longrightarrow S$ the Dirac operator. **Then** $D^2 = Đ + \operatorname{Sc} + \sigma(\omega)$**,** where $\sigma(\omega)$ denotes the Clifford multiplication by $\omega \in \Lambda^2(M)$.

Proof. Step 1: The computation is local, **hence we can assume that** $S = S_0 \otimes L$**,** where L is a line bundle, and S_0 the spinors. We represent a section of S by $s \otimes u$, where $s \in S_0$, and $u \in L$. Then

$$D^2(s \otimes u) = D^2(s) \otimes u + \sigma(s \otimes \nabla^2(u)) + 2\sigma(D(s) \otimes \nabla(u)),$$

where σ denotes the Clifford multiplication $\Lambda^i(M) \otimes S_0 \to S_0$.

The Bochner-Weitzenbŏck formula for Spin^c-manifolds (2)

Step 2: The term $\sigma(s\otimes\nabla^2(u))$ is equal to $\sigma(\omega)(s)+s\otimes Đ(u)$. Indeed, it is equal to the Clifford product $s_0\cdot\sum_{i,j}x_ix_j\nabla_{x_i}\nabla_{x_j}u$, but

$$\sum_{i,j}x_ix_j\nabla_{x_i}\nabla_{x_j}u=\sum_i\nabla_{x_i}\nabla_{x_i}u+\frac{1}{2}\sum_{i\neq j}(x_ix_j+x_jx_i)(\nabla_{x_i}\nabla_{x_j}+\nabla_{x_i}\nabla_{x_j})u+$$
$$+\frac{1}{2}(x_ix_j+x_jx_i)(\nabla_{x_i}\nabla_{x_j}-\nabla_{x_i}\nabla_{x_j})u=$$
$$=\sigma(\omega)(u)+\sum_i\nabla_{x_i}\nabla_{x_i}u=\sigma(\omega)(u)+Đ(u).$$

This takes care of the term related to the second derivative of u. It remains only to show that $D^2(s)\otimes u+\sigma(D(s)\otimes\nabla(u))-\operatorname{Sc}$ is the sum of all terms of $Đ(s\otimes u)$ which involve first and second derivatives of s.

Step 3: From the Bochner-Weitzenböck formula, we obtain $D^2(s)\otimes u = Đ(s)\otimes u+\operatorname{Sc}$. On the other hand, the term $\sigma(D(s)\otimes\nabla(u))$ is expressed through the Clifford multiplication as $\sum_i x_i\nabla_{x_i}(s)\otimes\sum x_i\nabla_{x_i}(u)$. Using $x_i\nabla_{x_i}(s)x_j\nabla_{x_j}(u)=-x_j\nabla_{x_i}(s)x_i\nabla_{x_j}(u)$ when $i\neq j$, we obtain

$$\sigma(D(s)\otimes\nabla(u))=\sum_i\nabla_{x_i}(s)\otimes\nabla_{x_i}(u)$$

and this gives the part of $Đ(s\otimes u)$ which is of order 1 in s and u. ■

Spin^c-structure on 4-manifolds

REMARK: Clearly, $\frac{SU(2)\times S^1}{\mathbb{Z}/2} = U(2)$. This gives a pair of natural homomorphisms $\operatorname{Spin}^c(4) = \frac{SU(2)\times SU(2)\times S^1}{\mathbb{Z}/2} \to U(2)$.

DEFINITION: The complex spinor representations $\mathcal{S}_+$ and $\mathcal{S}_-$ are representation of Spin^c obtained from these two homomorphisms and the fundamental representation of $U(2)$.

REMARK: Suppose that M is almost complex. Then $\mathcal{S}_+ \oplus \mathcal{S}_- = \Lambda^{*,0}(M) \otimes L$, where L is a line bundle. **The bundle $\mathcal{S}_+$ is identified with $L \oplus L \otimes \Lambda^{2,0}(M)$ and $\mathcal{S}_1$ with $L \otimes \Lambda^{1,0}(M)$.** Clifford multiplication flips the sign: $TM \otimes \mathcal{S}_\pm \to \mathcal{S}_\mp$.

DEFINITION: The map $q: \mathcal{S}_+ \otimes \overline{\mathcal{S}}_+ \to \Lambda^+(M)$ is defined as follows. Let $\sigma: \mathcal{S}_+ \otimes \Lambda^+(M) \to \mathcal{S}_+$ denote the Clifford multiplication map. Dualizing, we obtain a map $\hat{\sigma}: \mathcal{S}_+ \otimes \mathcal{S}_+ \to \Lambda^+(M)$. **We define $q(\psi) := \hat{\sigma}(\psi, \overline{\psi})$.**

The Seiberg-Witten equations

DEFINITION: Consider a Spin^c-structure on a compact orientable 4-manifold associated with a complex line bundle L, and let A denote the connection in L. Consider a spinor $\psi \in \mathcal{S}_+$. Let $F_A^+ \in \Lambda^+(M)$ denote the $+$-part of the curvature of A, and D_A the Dirac operator. The pair (A, ψ) **satisfies the Seiberg-Witten equations** if $D_A(\psi) = 0$ and $F_A^+ = q(\psi)$.

THEOREM: The space $\mathcal{M}$ of solutions of SW-equations up to the natural $U(1)$-action $\psi \to \lambda\psi$ **is compact.** It is **smooth and orientable** for general A when $b_2^+(M) > 0$. When $b_2^+(M) > 0$, the diffeomorphism type of $\mathcal{M}$ **is independent from the choice of A and the metric on the manifold** (as long as A remains generic). Finally,

$$\dim \mathcal{M} = b_1(M) - 1 - b_2^+(M) + \frac{c_1(L)^2 - \tau}{4},$$

where $\tau := b_2^+ - b_2^-$ is the signature of M.

Proof: *We will slowly approach the proof in the next lectures.*

REMARK: Compactness is spectacular and holds only for SW-equations. The rest of observations are straightforward and hold in some form for many other moduli spaces of geometric origin.