Seiberg-Witten invariants,

lecture 7: the Seiberg-Witten invariants

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The Bochner-Weitzenböck formula for Spin^c-manifolds (reminder)

THEOREM: Let M be a Riemannian manifold with Spin^c-structure, S the bundle of Spin^c-spinors, $Đ: S \longrightarrow S$ the rough Laplacian, ω the curvature of the corresponding complex line bundle, Sc the multiplication by the scalar curvature, and $D: S \longrightarrow S$ the Dirac operator. **Then $D^2 = Đ + \operatorname{Sc} + \sigma(\omega)$,** where $\sigma(\omega)$ denotes the Clifford multiplication by $\omega \in \Lambda^2(M)$.

Proof. Step 1: The computation is local, **hence we can assume that $S = S_0 \otimes L$,** where L is a line bundle, and S_0 the spinors. We represent a section of S by $s \otimes u$, where $s \in S_0$, and $u \in L$. Then

$$D^2(s \otimes u) = D^2(s) \otimes u + \sigma(s \otimes \nabla^2(u)) + 2\sigma(D(s) \otimes \nabla(u)),$$

where σ denotes the Clifford multiplication $\Lambda^i(M) \otimes S_0 \to S_0$.

The Bochner-Weitzenböck formula for Spin^c-manifolds (2): reminder

Step 2: The term $\sigma(s\otimes\nabla^2(u))$ is equal to $\sigma(\omega)(s)+s\otimes Ð(u)$. Indeed, it is equal to the Clifford product $s_0\cdot\sum_{i,j}x_ix_j\nabla_{x_i}\nabla_{x_j}u$, but

$$\sum_{i,j}x_ix_j\nabla_{x_i}\nabla_{x_j}u=\sum_i\nabla_{x_i}\nabla_{x_i}u+\frac{1}{2}\sum_{i\neq j}(x_ix_j+x_jx_i)(\nabla_{x_i}\nabla_{x_j}+\nabla_{x_i}\nabla_{x_j})u+$$
$$+\frac{1}{2}(x_ix_j+x_jx_i)(\nabla_{x_i}\nabla_{x_j}-\nabla_{x_i}\nabla_{x_j})u=$$
$$=\sigma(\omega)(u)+\sum_i\nabla_{x_i}\nabla_{x_i}u=\sigma(\omega)(u)+Ð(u).$$

This takes care of the term related to the second derivative of u. It remains only to show that $D^2(s)\otimes u+\sigma(D(s)\otimes\nabla(u))-\mathrm{Sc}$ is the sum of all terms of $Ð(s\otimes u)$ which involve first and second derivatives of s.

Step 3: From the Bochner-Weitzenböck formula, we obtain $D^2(s)\otimes u=Ð(s)\otimes u+\mathrm{Sc}$. On the other hand, the term $\sigma(D(s)\otimes\nabla(u))$ is expressed through the Clifford multiplication as $\sum_i x_i\nabla_{x_i}(s)\otimes\sum x_i\nabla_{x_i}(u)$. Using $x_i\nabla_{x_i}(s)x_j\nabla_{x_j}(u)=-x_j\nabla_{x_i}(s)x_i\nabla_{x_j}(u)$ when $i\neq j$, we obtain

$$\sigma(D(s)\otimes\nabla(u))=\sum_i\nabla_{x_i}(s)\otimes\nabla_{x_i}(u)$$

and this gives the part of $Ð(s\otimes u)$ which is of order 1 in s and u. ∎

Spin^c-structure on 4-manifolds (reminder)

REMARK: Clearly, $\frac{SU(2)\times S^1}{\mathbb{Z}/2} = U(2)$. This gives a pair of natural homomorphisms $\operatorname{Spin}^c(4) = \frac{SU(2)\times SU(2)\times S^1}{\mathbb{Z}/2} \to U(2)$.

DEFINITION: The complex spinor representations $\mathcal{S}_+$ and $\mathcal{S}_-$ are representation of Spin^c obtained from these two homomorphisms and the fundamental representation of $U(2)$.

REMARK: Suppose that M is almost complex. Then $\mathcal{S}_+ \oplus \mathcal{S}_- = \Lambda^{*,0}(M)\otimes L$, where L is a line bundle. **The bundle $\mathcal{S}_+$ is identified with $L \oplus L\otimes \Lambda^{2,0}(M)$ and $\mathcal{S}_1$ with $L\otimes\Lambda^{1,0}(M)$.** Clifford multiplication flips the sign: $TM\otimes\mathcal{S}_\pm \to \mathcal{S}_\mp$.

DEFINITION: The sesquilinear quadratic form $q: \mathcal{S}_+ \to \Lambda^+(M)$ is defined as follows. Let $\sigma: \mathcal{S}_+ \otimes \Lambda^+(M) \to \mathcal{S}_+$ denote the Clifford multiplication map. Dualizing, we obtain a map $\hat\sigma: \mathcal{S}_+ \otimes \mathcal{S}_+ \to \Lambda^+(M)$. **We define $q(\psi) := \hat\sigma(\psi,\overline{\psi})$.**

Clifford multiplication in $\mathbb{R}^4$: explicit calculations

Let e_1, e_3, e_3, e_4 be the orthonormal basis in $V = \mathbb{R}^4$, and $\omega_1 := e_1 \wedge e_2 + e_3 \wedge e_4$, $\omega_2 := e_1 \wedge e_3 - e_2 \wedge e_4$, $\omega_3 := e_1 \wedge e_4 + e_2 \wedge e_3$ the corresponding basis in $\Lambda^+(V)$. Consider the complex structure $I(e_1) = e_2$, $I(e_3) = e_4$. We identify $\mathcal{S}_+ = \mathbb{C}^2$ with the fundamental representation of $U(2)$ as above.

CLAIM: In these conventions, **the Clifford multiplication $\mathrm{Cl}(\omega_i)$ acts on $\mathcal{S}_+$ as**

$$\mathrm{Cl}(\omega_1) = \begin{pmatrix} -2\sqrt{-1} & 0 \\ 0 & -2\sqrt{-1} \end{pmatrix}, \mathrm{Cl}(\omega_2) = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \mathrm{Cl}(\omega_3) = \begin{pmatrix} 0 & -2\sqrt{-1} \\ -2\sqrt{-1} & 0 \end{pmatrix}. \quad (*)$$

Proof: By definition, $\mathrm{Cl}(\omega_i)$ act the same way as the matrices $2I, 2J, 2K \in \mathfrak{su}(2) = \mathfrak{so}(3) = \mathfrak{u}(1, \mathbb{H})$, act on the fundamental representation of $U(2)$, identified with $\mathbb{C}^2 = \mathcal{S}_+$. The matrices (*) are (up to a factor of 2) standard generators of $\mathfrak{su}(2)$. ■

CLAIM: Let $\psi = (a, b) \in \mathcal{S}_+$ be a spinor. **Then**

$$q(\psi) = (|a|^2 - |b|^2)\omega_1 + 2\,\mathrm{Im}(a\bar{b})\omega_2 + 2\,\mathrm{Re}(a\bar{b})\omega_3. \quad (**)$$

Proof: Clearly, $\psi \otimes \overline{\psi}^* = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} \bar{a} & \bar{b} \end{pmatrix}$ acts on $\mathcal{S}_+$ as $\begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix}$. Substituting the expressions for $\mathrm{Cl}(\omega_i)$, we obtain (**). ■

Quadratic form q: basic properties

CLAIM: Let $\psi \in \mathcal{S}_+$ be a spinor on a $V = \mathbb{R}^4$ and $\omega\psi$ the result of Clifford multiplication by a tensor $\omega \in \Lambda^2 V$. **Then $(\omega\psi, \overline{\psi}) = (\omega, q(\psi))$, where $(\cdot,\cdot)$ denotes the scalar product.**

Proof: By definition of q, the scalar product of $q(x)$ with $A \in \Lambda^2 M$ is equal to $(\mathrm{Cl}(A)x, \overline{x})$. ■

CLAIM: In these assumptions, $(q(\psi), q(\psi)) = |\psi|^4$.

Proof: $\psi \to (q(\psi), q(\psi))$ is $U(2)$-invariant positive function on $\mathbb{C}^2 = \mathcal{S}_+$, satisfying $(q(\lambda\psi), q(\lambda\psi)) = |\lambda|^4 (q(\psi), q(\psi))$. It is not hard to see that such a function is unique up to a scalar multiplier. The constant can be computed explicitly using the previous slide (but we don't care). ■

The Seiberg-Witten equations (reminder)

DEFINITION: Consider a Spin^c-structure on a compact orientable 4-manifold associated with a complex line bundle L, and let A denote the connection in L. Consider a spinor $\psi \in \mathcal{S}_+$. Let $F_A^+ \in \Lambda^+(M)$ denote the $+$-part of the curvature of A, and D_A the Dirac operator. The pair (A, ψ) **satisfies the Seiberg-Witten equations** if $D_A(\psi) = 0$ and $F_A^+ = q(\psi)$.

Theorem 1: The space $\mathcal{M}$ of solutions of SW-equations up to the natural $U(1)$-action $\psi \to \lambda\psi$ **is compact.** It is **smooth and orientable** for general A when $b_2^+(M) > 0$. When $b_2^+(M) > 0$, the diffeomorphism type of $\mathcal{M}$ **is independent from the choice of A and the metric on the manifold** (as long as A remains generic). Finally,

$$\dim \mathcal{M} = b_1(M) - 1 - b_2^+(M) + \frac{c_1(L)^2 - \tau}{4},$$

where $\tau := b_2^+ - b_2^-$ is the signature of M.

Proof: *We will slowly approach the proof in the next lectures.*

REMARK: Compactness is spectacular and holds only for SW-equations. The rest of observations are straightforward and hold in some form for many other moduli spaces of geometric origin.

The Seiberg-Witten equations: an a priori estimate

PROPOSITION: Let (A, ψ) be a solution of SW-equations, $\psi \neq 0$. **Then** $|\psi|^2 \leqslant \max_M(-\operatorname{Sc})$, where Sc denotes the scalar curvature.

Proof. Step 1: $D_A^2 = Đ_A + \operatorname{Sc} + \sigma(F_A)$. The $-$-part of F_A acts trivially on $\mathcal{S}_+$, hence $0 = Đ_A(\psi) + \operatorname{Sc}\psi + \sigma(F_A^+)\psi$. **Taking the scalar product with $\overline{\psi}$, we obtain $-(Đ_A\psi, \overline{\psi}) = \operatorname{Sc}|\psi|^2 + (\sigma(F_A^+)\psi, \overline{\psi})$.** The last term is equal to $\frac{q(\psi)\wedge F_A^+}{\operatorname{Vol}}$.

Step 2: Since $F_A^+ = q(\psi)$, we have $q(\psi) \wedge F_A^+ = |F_A^+|^2$. Then $-(Đ_A\psi, \overline{\psi}) = \operatorname{Sc}|\psi|^2 + |F_A^+|^2$. Let $x \in M$ be the point where $|\psi|$ reaches its maximum. At this point,

$$(Đ_A\psi, \overline{\psi}) = \operatorname{Tr}_{i,j}(\nabla_{x_i}\nabla_{x_j}\psi, \overline{\psi}) = \\ = \operatorname{Tr}_{i,j}\nabla_{x_i}\nabla_{x_j}|\psi|^2 + \operatorname{Tr}_{i,j}(\nabla_{x_i}\psi, \nabla_{x_j}\overline{\psi}) = \Delta|\psi|^2 + |\nabla\psi|^2.$$

Since $|\psi|^2$ is maximal at x, we have $\Delta|\psi|^2 \geqslant 0$. **Therefore, $(Đ_A\psi, \overline{\psi}) \geqslant 0$ at x.**

Step 3: The equation in Step 1, evaluated at x, gives $0 \geqslant -(Đ_A\psi, \overline{\psi}) = \operatorname{Sc}|\psi|^2 + |F_A^+|^2$. On the other hand, $|F_A^+|^2 = |q(\psi, \overline{\psi})|^2 = |\psi|^4$, hence **at x we have $|\psi|^4 \leqslant -\operatorname{Sc}|\psi|^2$, giving the estimate $|\psi|^2(x) = \max_M|\psi|^2 \leqslant -\operatorname{Sc}(x) \leqslant \max_M - \operatorname{Sc}$.** ■

More universal bounds

REMARK: For the sequel, another formula for $\dim \mathcal{M}$ is more appropriate; it is equivalent to the one given above:

$$\dim \mathcal{M} = b_1(M) - 1 - b_2^+(M) + \frac{c_1(L)^2 - \tau}{4} = \frac{1}{4}(c_1(L)^2 - 2\chi - 3\tau),$$

where τ is the signature of M and χ its Euler characteristic.
EXERCISE: Prove that these two numbers are equal.

PROPOSITION: Let (A, ψ) be a solution of SW-equations on (M, g, L), for A generic, and $k_- := \max_M(-\mathrm{Sc})$. **Then $|F_A^+| \leqslant k_-$, $\int_M |F_A^+|^2 \operatorname{Vol}_M \leqslant \int_M k_-^2 \operatorname{Vol}$ and $\int_M |F_A^-|^2 \operatorname{Vol}_M \leqslant \int_M k_-^2 \operatorname{Vol} - 8\pi^2\chi - 12\pi^2\tau$.**

Proof. Step 1: The first inequality was proven today. The second inequality is obtained by integrating the first. For the last inequality we use $\dim \mathcal{M} \geqslant 0$, becayse it is non-empty. **This gives $c_1(L)^2 - 2\chi - 3\tau \geqslant 0$.**

Step 2: Clearly, $c_1(L)^2 = \frac{1}{4\pi^2} \int_M F_A \wedge F_A = \frac{1}{4\pi^2} \int_M |F_A^+|^2 \operatorname{Vol} - \frac{1}{4\pi^2} \int_M |F_A^-|^2 \operatorname{Vol}$. Comparing with $c_1(L)^2 - 2\chi - 3\tau \geqslant 0$, and using $\int_M |F_A^-|^2 \operatorname{Vol}_M \leqslant \int_M k_-^2 \operatorname{Vol}$, we obtain

$$\int_M k_-^2 \operatorname{Vol} - 2\chi - 3\tau \geqslant \frac{1}{4\pi^2} \int_M |F_A^-|^2.$$

■

Topological implications of the universal bounds

Corollary 1: For generic A, **the space $\mathcal{M}$ is empty except for finitely many cohomology classes $c_1(L) \in H^2(M,\mathbb{Z})$.**

Proof: Let $H: \Lambda^2(M) \to \mathcal{H}^2(M)$ be the orthogonal projection to harmonic forms. Clearly, H is 1-Lipschitz with respect to the natural L^2-metric $\|\omega\| := \int_M \omega \wedge *\omega = \int_M (\omega^+)^2 - \int_M (\omega^-)^2$ on $\Lambda^2(M)$. Then

$$\int_M k_-^2 \operatorname{Vol} - 8\pi^2\chi - 12\pi^2\tau + \int_M k_-^2 \operatorname{Vol} \geqslant \|F_A^+\|^2 + \|F_A^-\|^2 \geqslant \|H(F_A)\|^2.$$

However, the form $x \to \|H(x)\|^2$ is positive definite on $H^2(M,\mathbb{R}) = \mathcal{H}^2(M)$, hence $c_1(L)$ **belongs to a ball of a fixed radius, depending on $\int_M k_-^2 \operatorname{Vol}$, τ and χ.** ■

The universal bundle

Let M be a compact 4-manifold, (L, A) a Spin^c-structure, and $\mathcal{M}$ the corresponding moduli of solutions of the SW-equations up to the natural $U(1)$-action $\psi \to \lambda\psi$.

CLAIM: For A generic, any solution (A, ψ) of SW-equation **satisfies $\psi \neq 0$.**

Proof. Step 1: Any anti-selfdual closed 2-form ω is harmonic. Indeed, $d^*\omega = - * d\omega = 0$.

Step 2: If $\psi = 0$, we have $F_A^+ = q(\psi) = 0$, hence the curvature of A is anti-selfdual. Let θ be a non-closed 1-form on M. Replacing the connection A by $A + \theta$ takes its curvature F_A to $F_A + d\theta$, **which breaks the anti-selfduality,** because $d\theta$ is exact, hence cannot be anti-selfdual. ■

DEFINITION: Let A be generic. **The universal bundle** over $\mathcal{M}_A$ is the line bundle associated with the principal $U(1)$-bundle of all ψ solving SW-equations.

The Seiberg-Witten invariant

DEFINITION: Let M be a compact oriented Riemannian 4-manifold with $b_2^+ > 0$, (L, A) a Spin^c-structure and $\mathcal{M}$ the space of solutions of SW-equation. **The Seiberg-Witten invariant** $SW(M, L)$ is a number, associated with M and L as follows. If $\dim \mathcal{M}$ is odd, we set $SW(M, L) = 0$. If it is even, $SW(M, L) := \int_{\mathcal{M}} c_1(U)^d$, where U is the universal bundle and $2d = \dim_{\mathbb{R}} \mathcal{M}$.

THEOREM: This number **is independent from the choice of the metric on M and A, if A is generic.** Moreover, $SW(M, L) \neq 0$ **for all $c_1(L)$ except finitely many.**

Proof: Theorem 1, Corollary 1. ∎