Seiberg-Witten invariants,

lecture 8: Seiberg-Witten invariants for Kähler surfaces

IMPA, sala 236

Misha Verbitsky, March 2, 2026, 17:00

`http://verbit.ru/IMPA/SW-2026/`

Some results about the Seiberg-Witten invariants

In Lecture 7, we defined **the Seiberg-Witten invariants** $P \to SW(P)$, integer invariants, associated to the Spin^c-structures.

THEOREM: (Taubes, Friedman-Morgan)
There is a blowup formula relating $SW(M\#\overline{\mathbb{C}P^2})$ and $SW(M)$: **we can always express $SW(M)$ through $SW(M\#\overline{\mathbb{C}P^2})$ and vice versa.**

THEOREM: (Taubes)
For any M, N with $b_+ > 0$, **one has $SW(M\#N) = 0$.**

COROLLARY: Let M be a K3 surface, and $M\#\overline{\mathbb{C}P^2}$ its blow-up. **Then $M\#\overline{\mathbb{C}P^2}$ is not diffeomorphic, but homeomorphic to $\#_3\mathbb{C}P^2\#_{19}\overline{\mathbb{C}P^2}$.**

Seiberg-Witten invariants and symplectic structures

The following theorems, relating SW-invariants to symplectic structures, are due to C. Taubes.

THEOREM: Let ω be a symplectic structure on a compact 4-manifold M. We denote $c_1(M)$ by c. Let P_0 denote the Spin^c-structure associated with the standard almost complex structure on M, and P_v a Spin^c-structure associated with $c_1(L) = v$, where L denotes the determinant class. **Let $c := c_1(M)$. Then $SW(P_0) = 1$, and $SW(P_c) = \pm 1$.** Moreover, **if $SW(v) \neq 0$, one has $\int_M v \wedge \omega \leqslant \int_M c \wedge \omega$.**

REMARK: Today **we will prove this theorem for Kähler surfaces.**

COROLLARY: $\#_m \mathbb{C}P^2$ has no symplectic structure when $m > 0$.

REMARK: For m odd, $\#_m \mathbb{C}P^2$ has no almost complex structures, and the obstruction is topological; **for m even, the only known obstruction comes from SW-invariants.**

Gromov invariants

Let M be an almost complex symplectic surface, with general almost complex structure, $v \in H^2(M,\mathbb{Z})$. Let $\mathcal{H}_v$ denote the set of all formal sums $\sum m_i C_i$, where C_i are smooth pseudoholomorphic curves, where $m_i > 0$ are integers, and the following properties are satisfied.

(i) $C_i \cap C_j = \emptyset$ for $i \neq j$.

(ii) Let $v_i \in H^2(M,\mathbb{Z})$ the Poincaré dual classes to $[C_i]$, and $c = c_1(M)$. Then $\sum_i m_i v_i = v$. (iii) (the dimension of the deformation space) $d_i := \frac{1}{2}(v_i \cdot v_i - c \cdot v_i) \geqslant 0$.

(iv) Suppose that $d := \frac{1}{2}(v \cdot v - c \cdot c) > 0$. Fix a set Ω of d generic points, $\Omega = \coprod \Omega_i$ with $|\Omega_i| = d_i$, and ask for C_i to contain Ω_i.

DEFINITION: We define an integer $Gr(v)$ **(the Taubes' Gromov invariant)** as follows. If $d < 0$, we set $Gr(v) = 0$. Otherwise, we define $Gr(v) := \sum_{h \in \mathcal{H}_v} \prod_i r(C_i, m_i)$, where $r(C_i, m_i) := \pm 1$ depending on the orientation defined by the almost complex structure on C_i.

THEOREM: If $b_2^+ > 0$, then $SW(P_v) = Gr(v)$.

Spin^c-structure on 4-manifolds (reminder)

REMARK: Clearly, $\frac{SU(2)\times S^1}{\mathbb{Z}/2} = U(2)$. This gives a pair of natural homomorphisms $\operatorname{Spin}^c(4) = \frac{SU(2)\times SU(2)\times S^1}{\mathbb{Z}/2} \to U(2)$.

DEFINITION: **The complex spinor representations** $\mathcal{S}_+$ and $\mathcal{S}_-$ are representation of Spin^c obtained from these two homomorphisms and the fundamental representation of $U(2)$.

REMARK: Suppose that M is almost complex. Then $\mathcal{S}_+ \oplus \mathcal{S}_- = \Lambda^{*,0}(M) \otimes L$, where L is a line bundle. **The bundle $\mathcal{S}_+$ is identified with $L \oplus L \otimes \Lambda^{2,0}(M)$ and $\mathcal{S}_1$ with $L \otimes \Lambda^{1,0}(M)$.** Clifford multiplication flips the sign: $TM \otimes \mathcal{S}_\pm \to \mathcal{S}_\mp$.

DEFINITION: The sesquilinear quadratic form $q: \mathcal{S}_+ \to \Lambda^+(M)$ is defined as follows. Let $\sigma: \mathcal{S}_+ \otimes \Lambda^+(M) \to \mathcal{S}_+$ denote the Clifford multiplication map. Dualizing, we obtain a map $\hat{\sigma}: \mathcal{S}_+ \otimes \mathcal{S}_+ \to \Lambda^+(M)$. **We define $q(\psi) := \hat{\sigma}(\psi, \overline{\psi})$.**

Clifford multiplication in $\mathbb{R}^4$: explicit calculations (reminder)

Let e_1, e_3, e_3, e_4 be the orthonormal basis in $V = \mathbb{R}^4$, and $\omega_1 := e_1 \wedge e_2 + e_3 \wedge e_4$, $\omega_2 := e_1 \wedge e_3 - e_2 \wedge e_4$, $\omega_3 := e_1 \wedge e_4 + e_2 \wedge e_3$ the corresponding basis in $\Lambda^+(V)$. Consider the complex structure $I(e_1) = e_2$, $I(e_3) = e_4$. We identify $\mathcal{S}_+ = \mathbb{C}^2$ with the fundamental representation of $U(2)$ as above.

CLAIM: In these conventions, **the Clifford multiplication $\mathrm{Cl}(\omega_i)$ acts on $\mathcal{S}_+$ as**

$$\mathrm{Cl}(\omega_1) = \begin{pmatrix} -2\sqrt{-1} & 0 \\ 0 & 2\sqrt{-1} \end{pmatrix}, \mathrm{Cl}(\omega_2) = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \mathrm{Cl}(\omega_3) = \begin{pmatrix} 0 & -2\sqrt{-1} \\ -2\sqrt{-1} & 0 \end{pmatrix}. \quad (*)$$

Proof: By definition, $\mathrm{Cl}(\omega_i)$ act the same way as the matrices $2I, 2J, 2K \in \mathfrak{su}(2) = \mathfrak{so}(3) = \mathfrak{u}(1, \mathbb{H})$, act on the fundamental representation of $U(2)$, identified with $\mathbb{C}^2 = \mathcal{S}_+$. The matrices (*) are (up to a factor of 2) standard generators of $\mathfrak{su}(2)$. ■

CLAIM: Let $\psi = (a, b) \in \mathcal{S}_+$ be a spinor. **Then**

$$q(\psi) = (|a|^2 - |b|^2)\omega_1 + 2\,\mathrm{Im}(a\bar{b})\omega_2 + 2\,\mathrm{Re}(a\bar{b})\omega_3. \quad (**)$$

Proof: Clearly, $\psi \otimes \overline{\psi}^* = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} \overline{a} & \overline{b} \end{pmatrix}$ acts on $\mathcal{S}_+$ as $\begin{pmatrix} |a|^2 & a\bar{b} \\ \overline{a}b & |b|^2 \end{pmatrix}$. Substituting the expressions for $\mathrm{Cl}(\omega_i)$, we obtain (**). ■

Quadratic form q: basic properties (reminder)

CLAIM: Let $\psi \in \mathcal{S}_+$ be a spinor on a $V = \mathbb{R}^4$ and $\omega\psi$ the result of Clifford multiplication by a tensor $\omega \in \Lambda^2 V$. **Then $(\omega\psi, \overline{\psi}) = (\omega, q(\psi))$, where $(\cdot, \cdot)$ denotes the scalar product.**

Proof: By definition of q, the scalar product of $q(x)$ with $A \in \Lambda^2 M$ is equal to $(\mathrm{Cl}(A)x, \overline{x})$. ∎

CLAIM: In these assumptions, $(q(\psi), q(\psi)) = |\psi|^4$.

Proof: $\psi \to (q(\psi), q(\psi))$ is $U(2)$-invariant positive function on $\mathbb{C}^2 = \mathcal{S}_+$, satisfying $(q(\lambda\psi), q(\lambda\psi)) = |\lambda|^4 (q(\psi), q(\psi))$. It is not hard to see that such a function is unique up to a scalar multiplier. The constant can be computed explicitly using the previous slide (but we don't care). ∎

The Seiberg-Witten equations (reminder)

DEFINITION: Consider a Spin^c-structure on a compact orientable 4-manifold associated with a complex line bundle L, and let A denote the connection in L. Consider a spinor $\psi \in \mathcal{S}_+$. Let $F_A^+ \in \Lambda^+(M)$ denote the $+$-part of the curvature of A, and D_A the Dirac operator. The pair (A, ψ) **satisfies the Seiberg-Witten equations** if $D_A(\psi) = 0$ and $F_A^+ = q(\psi)$.

Theorem 1: The space $\mathcal{M}$ of solutions of SW-equations up to the natural $U(1)$-action $\psi \to \lambda\psi$ **is compact.** It is **smooth and orientable** for general A when $b_2^+(M) > 0$. When $b_2^+(M) > 0$, the diffeomorphism type of $\mathcal{M}$ **is independent from the choice of A and the metric on the manifold** (as long as A remains generic). Finally,

$$\dim \mathcal{M} = b_1(M) - 1 - b_2^+(M) + \frac{c_1(L)^2 - \tau}{4},$$

where $\tau := b_2^+ - b_2^-$ is the signature of M.

Proof: *We will slowly approach the proof in the next lectures.*

REMARK: Compactness is spectacular and holds only for SW-equations. The rest of observations are straightforward and hold in some form for many other moduli spaces of geometric origin.

Spin^c-structures on a Kähler 4-manifold

Let (M,ω) be a Kähler surface (that is, 4-manifold). As explained in Lecture 7, the complex spinor bundles are $\mathcal{S}_+ := L_0 \otimes (\Lambda^0(M) \oplus \Lambda^{0,2}(M))$ and $\mathcal{S}_+ := L_0 \otimes \Lambda^{0,1}(M)$, where $L := L_0^2 \otimes K_M^{-1}$ is the determinant line bundle of the Spin^c-structure. The Dirac operator is $(\partial + \partial^*) \otimes \nabla_{A_0}$, where ∇_{A_0} denotes the connection on L_0. Note that L_0 **is not assumed to be holomorphic**, **its curvature does not need to be of type (1,1).**

Let $\psi \in \mathcal{S}_+$ be a spinor, $\psi = (a,b) \in= L_0 \otimes (\Lambda^0(M) \oplus \Lambda^{0,2}(M))$ Then $\psi \otimes \overline{\psi} = \begin{pmatrix} |a|^2 & a\overline{b} \\ \overline{a}b & |b|^2 \end{pmatrix}$. Identifying $\mathfrak{su}(2)$ with $\Lambda^+(M) = \mathbb{R}\omega \oplus \operatorname{Re}(\Lambda^{2,0}(M))$, we obtain $q(\psi) = (|b|^2 - |a|^2)\omega - a\overline{b} + \overline{a}b$.

Solutions of SW-equations on a Kähler surface

CLAIM: The SW-equations applied to $\psi = (a, b)$ are expressed as

$$\begin{aligned}&\overline{\partial}_{A_0}(a) + \overline{\partial}^*_{A_0}(b) = 0\\ &F_A^{1,1} = -|a|^2 + |b|^2, \text{ and } \ F_A^{0,2} = \overline{a}b, \quad (***)\end{aligned}$$

where $\overline{\partial}_{A_0} : \Lambda^{p,q}(M) \otimes L_0 \to \Lambda^{p,q+1}(M) \otimes L_0$ denotes the (0,1)-part of the L_0-valued differential $d_{A_0} : \Lambda^i(M) \otimes L_0 \to \Lambda^{i+1}(M) \otimes L_0$ on L_0-valued forms, and $\overline{\partial}^*_{A_0}$ its Hermitian dual.

LEMMA: Let (M, ω) be a Kähler surface. **Then for any solution of (***), we have $a\overline{b} = 0$.**

Proof: Applying $\overline{\partial}_{A_0}$ to the first equation of (***), we obtain

$$\overline{\partial}_{A_0}\overline{\partial}_{A_0}(a) + \overline{\partial}_{A_0}\overline{\partial}^*_{A_0}(b) = 0. \quad (+)$$

Clearly, $\overline{\partial}_{A_0}\overline{\partial}_{A_0}(a) = F^{0,2}_{A_0} \cdot a$. Since $L = L_0^2 \otimes K_M$, the (2,0)-part of the curvature of L_0 is $\frac{1}{2}F_A^+ = \overline{a}b$. Plugging it into (+), we obtain $\frac{1}{2}|a|^2 b = -\overline{\partial}_{A_0}\overline{\partial}^*_{A_0}(b)$. Multiplying both sides by $\overline{b}$ and integrating, we obtain

$$\int_M \frac{1}{2}|a|^2|b|^2 \operatorname{Vol} = -\int_M \|\overline{\partial}^*_{A_0}(b)\|^2 \operatorname{Vol}, \quad (++)$$

hence $\overline{\partial}^*_{A_0}(b) = 0$ and $ab = 0$. ■

Solutions of SW-equations on a Kähler surface (2)

COROLLARY: For any solution of SW-equations (***) over a Kähler surface, **one has $F_A^{0,2} = 0$.** *Ipso facto*, L and L_0 are holomorphic line bundles with curvature of type (1,1). **Moreover, $\overline{\partial}^*_{A_0}(b) = 0$.**

Proof: $F_A^{0,2} = a\bar{b} = 0$, as shown above. The second assertion follows from (++). ■

COROLLARY: $\bar{b}$ is a holomorphic section of $K_M \otimes L_0^{-1}$.

Proof: Kodaira relations give $\overline{\partial}^*_{A_0} = \sqrt{-1}\,[\Lambda, \partial_{A_0}]$. Applying this to b, we obtain $\Lambda\partial_{A_0}\beta = 0$. However, $\Lambda : \Lambda^{2,1}(M) \to \Lambda^{1,0}(M)$ is an isomorpism. Therefore, $\partial_{A_0} b = 0$, **hence $\overline{\partial}_{A_0}\bar{b} = 0$ and $\bar{b}$ is a holomorphic section of $K_M \otimes \overline{L}_0 = K_M \otimes L_0^{-1}$.** ■

COROLLARY: In these assumptions, **either a or b vanishes identically.**

Proof: If a is non-zero in an open subset of M, $b = 0$ in this subset. Since $\bar{b}$ is holomorphic, this implies that $b = 0$ everywhere on M. ■

Solutions of SW-equations on a Kähler surface are holomorphic

DEFINITION: Recall that **the degree** of a holomorphic line bundle L on a Kähler manifold (M, ω) is $\int_M c_1(L) \wedge \omega^{\dim_{\mathbb{C}} M - 1}$.

PROPOSITION: Let (a, b) be a non-zero solution of the Seiberg-Witten equation (***) on a Kähler surface. Then the connection ∇_A defines a holomorphic structure on L. Moreover, **either** $\deg L < 0$**, and** a **is an holomorphic section of** L_0**,** or $\deg L > 0$ **and** $\bar{b}$ **is a holomorphic section of** $K_M \otimes L_0$**.**

Proof: If $b = 0$, (***) implies $\bar{\partial}_{A_0}(a) = 0$, hence a is holomorphic. In this case $F_A^+ = -|a|^2\omega$, hence $\deg L < 0$. Otherwise $\bar{b}$ is holomorphic, and $F_A^+ = |b|^2\omega$ implies $\deg L > 0$. ■

Moduli of solutions of SW-equations on a Kähler surface

DEFINITION: A **Chern connection** on a Hermitian holomorphic bundle is a unitary connection ∇ which satisfies $\nabla^{0,1} = \bar\partial$.

THEOREM: For any Hermitian holomorphic bundle **the Chern connection exists, and is unique.** ■

REMARK: Since $GL(n,\mathbb{C})$ acts transitively on the set of all Hermitian metrics, **the gauge group $\operatorname{Aut}_{C^\infty_{\mathbb{C}}(L)}(L)$ acts transitively on the set of all Chern connections.**

The following corollary describes the moduli of solutions of SW-equations on a Kähler surface.

Corollary 1: Let (M,ω) be a Kähler surface, L the determinant bundle of a Spin^c-structure, $L = K_M \otimes L_0^{\otimes 2}$. Then:

(A) If $\deg L < 0$, solutions of SW-equations **are in bijective correspondence with the triples (holomorphic line bundle L_0, a Hermitian metric on L_0, a holomorphic section $a \in H^0(M, L_0)$ such that $F_A^+ = -|a|^2\omega$.**

(B) If $\deg L > 0$, solutions of SW-equations **are in bijective correspondence with the triples (holomorphic line bundle L_0, a Hermitian metric on L_0, a holomorphic section $\bar b \in H^0(M, L_0^{-1} \otimes K_M)$ such that $F_A^+ = |b|^2\omega$.**

■

REMARK: When $\psi = 0$, SW-equation is reduced to $F_A^+ = 0$. Therefore, when $\deg L = 0$, the moduli of solutions of SW-equations **are in bijective correspondence with ASD (anti-selfdual) connections on L.**

The connection A is uniquely determined by the holomorphic data

Proposition 1: Consider a Kähler surface, equipped with a Spin^c-structure. Fix a holomorphic structure on L_0, and let $(A, \psi = (a, b))$ be the solution of SW-equations. Then **the Chern connection A_0 on L_0 is uniquely determined by the conditions $F_A^+ = |b|^2\omega$ and $F_A^+ = -|a|^2\omega$.**

Proof. Step 1: Fix a Hermitian metric h on L_0, and let $h' = e^\lambda h$ be another metric, with A' the corresponding connection on L. Then $F_{A'} = F_A + \frac{1}{2} dd^c\lambda$, which gives $F_{A'}^+ = F_A + \Delta\lambda\omega$, where $\Delta = \frac{1}{2}(dd^c\lambda, \omega)$ is the Laplacian. The equation $F_{A'}^+ = -|a|^2_{h'}\omega$ becomes $F_A^+ + +\Delta\lambda\omega = -|a|^2_h e^\lambda\omega$. This is equivalent **to the following elliptic equation $\Delta\lambda = f - e^\lambda g$,** where f, g are smooth functions on M, $g \geqslant 0$, $g > 0$ almost everywhere, and $\int_X f\,\mathrm{Vol} > 0$.

The Kazdan-Warner equation

Step 2: The Kazdan-Warner equation $\Delta\lambda = f - e^{\lambda}g$ on a compact orientable Riemannian manifold M has a unique solution. Uniqueness is clear: let φ, ψ be solutions, $w = \psi - \varphi$ its difference, then Kazdan-Warner gives

$$0 \leqslant \int_M |dw|^2 \operatorname{Vol} = \int_M w\Delta w \operatorname{Vol} = -\int_M (\psi - phi)(e^{\psi} - e^{\varphi})g \operatorname{Vol},$$

a contradiction because $(\psi-\varphi)(e^{\psi}-e^{\varphi}) \geqslant 0$. Existence is slightly more tricky, J. L. Kazdan, F. W. Warner, *Curvature functions for compact 2-manifolds,* Annals of Math., Vol. 99, No. 1, 1974, pp. 14-47. ■

We have shown that the solution A of $F_A^+ = -|a|^2\omega$ and, similarly, of $F_A^+ = |b|^2\omega$ **is uniquely determined by the holomorphic data.** Therefore, Corollary 1 gives

COROLLARY: The moduli of non-reducible solutions of SW-equation **are in bijective correspondence with** $\mathbb{P}H^0(M, L_0)$ **when** $\deg L < 0$ **and with** $\mathbb{P}H^0(M, K_M \otimes L_0^{-1})$ **when** $\deg L > 0$**.** ■

SW-equations and degree of K_M

DEFINITION: A solution (ψ, A) of a SW-equation is called **reducible** when $\psi = 0$.

CLAIM: SW-equation **has no non-reducible solutions when** $\deg K_M < 0$**.**

Proof. Step 1: If $a \neq 0$, the bundle L_0 has a holomorphic section, hence $\deg L_0 > 0$. However, $L_0^2 \otimes K_M^{-1} = L$, hence $\deg L > 0$, which contradicts with Corollary 1.

Step 2: If $b \neq 0$, then $\bar{b} \in H^0(K_M \otimes L_0^{-1})$ is a non-zero holomorphic section, hence $\deg K_M - \deg L_0 > 0$, hence $\deg L_0 < \deg K_M < 0$. By Corollary 1, $\deg L > 0$, giving $2 \deg L_0 = \deg L - \deg K_M > 0$, a contradiction. ■

CLAIM: Consider the Spin^c-structure on a Kähler surface with L_0 a trivial bundle, and $\deg K_M > 0$. **Then the moduli space of solutions of the SW-equation is a single point.**

Proof: Since L_0 is trivial, $L = K_M^{-1}$, hence it has negative degree. Then $b = 0$ and a is a holomorphic section of L_0. Such section is unique up to $\mathbb{C}^*$-action on L_0. The constrain $F_A^+ = -|a|^2 \omega$ determines A uniquely by Kazdan-Warner (Proposition 1). ■

The universal bundle (reminder)

Let M be a compact 4-manifold, (L, A) a Spin^c-structure, and $\mathcal{M}$ the corresponding moduli of solutions of the SW-equations up to the natural $U(1)$-action $\psi \to \lambda\psi$.

CLAIM: For A generic, any solution (A, ψ) of SW-equation **satisfies $\psi \neq 0$.**

Proof. Step 1: Any anti-selfdual closed 2-form ω is harmonic. Indeed, $d^*\omega = - * d\omega = 0$.

Step 2: If $\psi = 0$, we have $F_A^+ = q(\psi) = 0$, hence the curvature of A is anti-selfdual. Let θ be a non-closed 1-form on M. Replacing the connection A by $A + \theta$ takes its curvature F_A to $F_A + d\theta$, **which breaks the anti-selfduality,** because $d\theta$ is exact, hence cannot be anti-selfdual. ■

DEFINITION: Let A be generic. **The universal bundle** over $\mathcal{M}_A$ is the line bundle associated with the principal $U(1)$-bundle of all ψ solving SW-equations.

The Seiberg-Witten invariant (reminder)

DEFINITION: Let M be a compact oriented Riemannian 4-manifold with $b_2^+ > 0$, (L, A) a Spin^c-structure and $\mathcal{M}$ the space of solutions of SW-equation. **The Seiberg-Witten invariant** $SW(M, L)$ is a number, associated with M and L as follows. If $\dim \mathcal{M}$ is odd, we set $SW(M, L) = 0$. If $\dim \mathcal{M} = 0$, $SW(M, L)$ is the number of solutions of SW-equations, counted with the sign which is determined by the orientation. If it is even and positive, $SW(M, L) := \int_{\mathcal{M}} c_1(U)^d$, where U is the universal bundle and $2d = \dim_{\mathbb{R}} \mathcal{M}$.

THEOREM: This number **is independent from the choice of the metric on M and A, if A is generic.** Moreover, $SW(M, L) \neq 0$ **for all $c_1(L)$ except finitely many.**

Proof: Lecture 7. ■

The standard involution on the set of Spin^c-structures

DEFINITION: Let $P_{SO(TM)\times S^1}$ be the principal bundle associated with the tangent bundle, and $\tilde P_{SO(TM)\times S^1}$ its double cover, which is fiberwise homeomorphic to Spin^c. The group $\mathbb{Z}/2$ acts on Spin$^c = \frac{SU(2)\times SU(2)\times S^1}{\pm 1}$ by $(a,b,t)\to(a,b,t^{-1})$, taking the determinant line bundle L to its complex conjugate. **We denote this infolution by $P\to -P$.**

REMARK: The involution $P\to -P$ **takes the solutions of SW-equation to solutions of SW-equation,** but it does not necessarily preserve the orientation on the moduli of solutions, hence $SW(P)=\pm SW(-P)$.

REMARK: The sign is equal to $(-1)^{\frac{1+b_2^+-b_1}{2}}$. For the proof, see Morgan's textbook, Corollary 6.8.4.

Dimension formula for almost complex manifolds

The dimension formula for the moduli of solutions of SW (Lecture 7):

$$\dim \mathcal{M} = b_1(M) - 1 - b_2^+(M) + \frac{c_1(L)^2 - \tau}{4} = \frac{1}{4}(c_1(L)^2 - 2\chi - 3\tau),$$

On the other hand, $p_1(M) = c_1(M)^2 - 2c_2(M)$ and $p_1(M) = 3\tau$. This gives $c_1(M)^2 = 2\chi + 3\tau$. Substituting $c_2(M) = \chi$, we obtain **the dimension formula**

$$\dim \mathcal{M} = c_1(L)^2 - c_1(M)^2.$$

SW-invariants for minimal Kähler surfaces

DEFINITION: A complex surface M is called **minimal** if M does not contain rational curves with self-intersection -1. It is called **of general type** if $\int_M c_1(M)^2 > 0$ and $\deg K_M > 0$.

THEOREM: Let M be a minimal surface of general type, and P a Spin^c-structure. **Then $SW(P) = 1$ if $P = P_I$ is the standard Spin^c-structure, $\mathcal{S} = \Lambda^0(M) \oplus \Lambda^{0,2}(M)$, $SW(-P) = \pm 1$, and $SW(P) = 0$ otherwise.**

Proof. Step 1: If $\deg K_M \leqslant 0$, Hodge index theorem would imply that $\int_M c_1(M)^2 \leqslant 0$. Since M is of general type, this is impossible, and $\deg K_M > 0$. Then $SW(P_I) = 1$ by Proposition 1, and $SW(-P_I) = \pm SW(P_I) = 1$.

Step 2: It remains only to show that $SW(P) = 0$ when $P \neq \pm P_I$. Let P be a Spin^c-structure, L is determinant bundle. Dimension formula gives $c_1(L)^2 \geqslant K_M^2 \geqslant 0$. Replacing L by a complex conjugate if necessary, we can always assume that $\deg L < 0$. A solution of SW-equations implies that for some holomorphic structure on L the bundle $L_0 = \sqrt{K_M \otimes L}$ has a non-zero holomorphic section.

SW-invariants for minimal Kähler surfaces (2)

Step 3: Since M is minimal, $(K_M, D) \geqslant 0$ for any effective divisor. Since $K_M \otimes L = L_0^{\otimes 2}$ is effective, we have $(K_M, K_M + L) \geqslant 0$. Since $\deg K_M > 0$ and $\deg L < 0$, there exists $t \in [0, 1]$ such that $(\omega, K_M + tL) = 0$. Hodge index theorem implies that $0 \geqslant (K_M + tL)^2$. The minimum of function $t \to (K_M + tL)^2$ occurs when $t = \frac{-(K_M,L)}{L^2}$, and it is equal to $K_M^2 - \frac{(K_M,L)}{L^2}$. Since $L^2 \geqslant K_M^2 \geqslant -(K_M, L)$, and $K_M^2 \geqslant 0$, we obtain that $K_M^2 - \frac{(K_M,L)}{L^2} \geqslant 0$. Therefore, $(K_M + L)^2 = 0$, and $(\omega, K_M + L) = 0$. By Hodge index theorem, this implies that $K_M + L$ is a torsion class; since $K_M + L$ has sections, it vanishes.

Step 4: This proves that L_0 is holomorphically trivial, hence $P = P_I$. ■

COROLLARY: Let M be a minimal surface of general type, and $f : M \to M$ an orientation-preserving diffeomorphism. **Then $f^*(c_1(M)) = \pm c_1(M)$.**