

Complex surfaces, home assignment 2: Cohomology of local systems

Rules: This is a class assignment for this week, for discussion in class Wednesday next week.

Remark 2.1. Throughout this assignment, all manifolds are assumed to be connected.

Exercise 2.1. Let $\lambda > 0$ be a real number. Define **weight λ homogeneous forms** on $\mathbb{R}^n \setminus 0$ as differential forms η which satisfy $\rho_t^* \eta = t^\lambda \eta$, where ρ_t is a homothety map $z \rightarrow tz$, $t \in \mathbb{R}$. Prove that a closed weight λ form is a differential of an weight λ form any $\lambda \neq 1$.

Hint. Use Cartan's formula $\text{Lie}_v(\alpha) = di_v \alpha + i_v d\alpha$.

Exercise 2.2. Let $M = \mathbb{R}^n \setminus 0 / (x \sim 2x)$ be a Hopf manifold, θ a closed, non-exact 1-form, $d_\theta = d + \theta$, and $H_\theta^*(M)$ cohomology of the complex $\Lambda^*(M), d_\theta$ ("Morse-Novikov cohomology"). Prove that $H_\theta^i(M) = 0$ for all i .

Hint. Use the previous exercise.

Exercise 2.3. Let θ be an exact 1-form on a manifold M . Construct an isomorphism between the complexes $(\Lambda^i(M), d_\theta)$ and $(\Lambda^i(M), d)$.

Exercise 2.4. Let θ be a closed 1-form on a connected manifold. Prove that $H_\theta^0(M) \neq 0$ if and only if θ is exact.

Definition 2.1. A **complex of vector spaces** is a collection of $\{A_i, i \in \mathbb{Z}\}$ of vector spaces equipped with the **differential** $d: A_i \rightarrow A_{i+1}$ such that $d^2 = 0$; its **cohomology groups** are $\frac{\ker d}{\text{im } d}$. **Morphism of complexes** $\phi: (A_*, d_A) \rightarrow (B_*, d_B)$ is a collection of homomorphisms $\phi_i: A_i \rightarrow B_i$ commuting with the differentials. **The cone** of such a morphism is a complex $C_i := B_i \oplus A_{i+1}$ with the differential $d_B + (-1)^i d_A + \phi: B_i \oplus A_{i+1} \rightarrow B_{i+1} \oplus A_{i+2}$.

Exercise 2.5. Let $A_* \xrightarrow{\phi} B_*$ be a morphism of complexes of vector spaces, and C_* its cone. Construct a long exact sequence

$$\dots \rightarrow H^{i-1}(C_*) \rightarrow H^i(A_*) \rightarrow H^i(B_*) \rightarrow H^i(C_*) \rightarrow H^{i+1}(A_*) \rightarrow \dots$$

Exercise 2.6. Let X be a manifold, $M := X \times S^1$, and $\pi: M \rightarrow S^1$ the standard projection. Denote by $T_X^1(M)$ the bundle of vectors tangent to the fibers of π , $\Lambda_X^1(M)$ its dual, and $\Lambda_X^*(M)$ the corresponding Grassmann algebra. Consider the de Rham differential d_X acting on $\Lambda_X^*(M)$ fiberwise along X .

- a. Prove that the cohomology of $(\Lambda_X^*(M), d_X)$ is isomorphic to $H^*(X) \otimes C^\infty S^1$.
- b. Let $\frac{d}{dt}$ be the derivative along the circle. Prove that $\frac{d}{dt}$ commutes with d_X .
- c. Let $C_0^\infty S^1$ denote the functions which average to zero. Prove that $\frac{d}{dt}$ is invertible on $C_0^\infty S^1$.
- d. Prove that the kernel and the cokernel of $H^*(X) \otimes C^\infty S^1 \xrightarrow{d/dt} H^*(X) \otimes C^\infty S^1$ are naturally isomorphic to $H^*(X)$.
- e. Prove that the map $\frac{d}{dt} + \text{Id} : C^\infty S^1 \rightarrow C^\infty S^1$ is an isomorphism.

Exercise 2.7. We work in assumptions of the previous exercise.

- a. Prove that $\Lambda^*(M)$ is (as a graded vector bundle) naturally isomorphic to $\Lambda_X^*(M) \oplus \Lambda_X^*(M) \wedge dt$, where dt is the constant coordinate 1-form on S^1 lifted to M .
- b. Let $\Phi : \Lambda_X^*(M) \rightarrow \Lambda_X^*(M) \wedge dt$ denote the operator $\alpha \mapsto \frac{d}{dt}\alpha \wedge dt$. Prove that

$$\Phi : (\Lambda_X^*(M), d_X) \rightarrow (\Lambda_X^*(M) \wedge dt, d_X)$$

is a morphism of complexes, and the de Rham algebra of M is naturally isomorphic to its cone.

- c. Let $\theta := dt$, and let $d_\theta : \Lambda^*(M) \rightarrow \Lambda^*(M)$ be the corresponding Morse-Novikov differential. Prove that $(\Lambda^*(M), d_\theta)$ is isomorphic to the cone of $\Phi + \mathbb{I}$, where $\mathbb{I}(\alpha) = \alpha \wedge dt$.
- d. Using the previous exercise, prove that Φ acts on the cohomology $H^*(X) \otimes C^\infty S^1$ of $(\Lambda_X^*(M), d_X)$ as $\frac{d}{dt}$, and $\Phi + \mathbb{I}$ acts as an isomorphism.

Exercise 2.8. Let X be a compact manifold, and $M = X \times S^1$, $\pi : M \rightarrow S^1$ the standard projection, $\theta := \pi^* dt$. Prove that $H_\theta^i(M) = 0$ for all i .

Hint. Use the previous exercise.