Complex surfaces, home assignment 3: Harmonic functions

Rules: This is a class assignment for this week, for discussion in class Wednesdays and Fridays after the lecture.

Exercise 3.1. Let $R = \mathbb{C}[t_1, ..., t_n]$ be a polynomial algebra, and $\operatorname{Diff}^* R \subset \operatorname{End}_{\mathbb{C}} R$ its algebra of differential operators, generated by multiplication with t_i and the derivation operators $\frac{\partial}{\partial t_i}$. Prove that this algebra is simple (has no non-trivial two-sided ideals).

Definition 3.1. Let M be a Riemannian manifold. The **Laplacian** on differential forms is $\Delta := dd^* + d^*d$. Restricted to functions, this gives $d^*df = \sum g^{ij} \frac{d^2}{dx_i dx_j}$, where x_i are coordinates, and g^{ij} the Riemannian form written in the basis dx_i, dx_j. A function f is **harmonic** if $\Delta f = 0$.

Exercise 3.2. Let η be a harmonic form with compact support on $\mathbb{R}^n$. Prove that $\eta = 0$.

Exercise 3.3. Let M be a compact n-manifold, $\alpha \in \Lambda^p M$, and $\alpha \in \Lambda^{n-p-1} M$. Prove that $\int d\alpha \wedge \beta = -(-1)^p \int \alpha \wedge d\beta$.

Exercise 3.4. Let M be a compact complex n-manifold, $\alpha \in \Lambda^p M$, and $\alpha \in \Lambda^{2n-p-1} M$. Prove that $\int \partial\alpha \wedge \beta = -(-1)^p \int \alpha \wedge \partial\beta$, where ∂ is the corresponding Dolbeault differential, that is, the (1,0)-part of de Rham differential.

Hint. Use the previous exercise and express ∂ through d and $d^c = IdI^{-1}$.

Exercise 3.5. Let f be a function on a Riemannian manifold M, and $X \subset M$ an oriented submanifold of codimension 1. Let ν be the normal vector field to X. Prove that $*df\Big|_X = \operatorname{Lie}_\nu(f) \operatorname{Vol}_X$.

Exercise 3.6. Let M be a compact Riemannian manifold with boundary, and u, v smooth functions on M. Prove that

$$\int u\Delta v \operatorname{Vol}_M + (-1)^{n-1} \int_M g(du, dv) \operatorname{Vol}_M = \int_{\partial M} u \operatorname{Lie}_\nu(v) \operatorname{Vol}_{\partial M}, \quad (3.1)$$

where ν is the unit vector field normal to the boundary.

Hint. Using the Stokes' formula, obtain

$$\int_{\partial M} u * dv = \int_M du \wedge *dv + \int_M u \cdot d{*}dv.$$

Use the previous exercise to express $\int_{\partial M} u * dv$ through the normal vector field.

Exercise 3.7. (Green representation formula)
In assumptions of the previous exercise, prove that

$$\int_M u\Delta v \operatorname{Vol}_M - \int_M v\Delta u \operatorname{Vol}_M = \int_{\partial M} (u \operatorname{Lie}_\nu v - v \operatorname{Lie}_\nu u) \operatorname{Vol}_{\partial M}$$

Hint. Substract (3.1) from the same relation exchanging u and v.

Exercise 3.8. Let $d_z \in C^\infty M$ be a Riemannian distance to z (it is smooth on a small neighbourhood of z), and $\nu := \mathsf{grad}(d_z)$ its gradient. Prove that any harmonic function f on M satisfies

$$\int_{\partial B_r} \operatorname{Lie}_\nu f \operatorname{Vol}_{\partial B_r} = 0.$$

where B_r is a ball of radius r.

Hint. Apply the Green representation to formula $v = 1$, $u = f$.

Exercise 3.9. Prove that the average of a harmonic function f in a ball $B_r(x)$ centered in x is equal to $f(x)$.

Exercise 3.10. Let f be a function on $\mathbb{R}^n\backslash 0$ with the standard flat metric which is bounded and harmonic. Prove that f can be extended to 0 smoothly.

Hint. Use the previous exercise.