Complex surfaces, home assignment 4: Fubini-Study form and its potential

Rules: This is a class assignment for this week, for discussion in class Wednesdays and Fridays after the lecture.

Exercise 4.1 (*). Let f a non-constant plurisubharmonic function on $\mathbb{C}$. Prove that f cannot be bounded.

Exercise 4.2. Let $z_1, ..., z_n$ be the standard coordinates on $\mathbb{C}^n$, and $l(z_1, ..., z_n) := \sum |z_i|^2$. Prove that $dd^c \log l = \frac{dd^c l}{l} - \frac{dl \wedge d^c l}{l^2}$ on $\Omega = \mathbb{C}^n \backslash 0$. Prove that the Hermitian form $dd^c \log l$ is degenerate, and can be written as $(-\sqrt{-1}) \sum_{i=1}^{n-1} \alpha_i \xi_i \wedge \bar{\xi}_i$, where $\xi_1, ..., \xi_n$ is a local frame in $\Lambda^{1,0}(\Omega)$, and α_i are positive functions.

Exercise 4.3. a. Consider the function $|z|^2 = z\bar{z}$ on $\mathbb{C}^*$, and let $\rho = z\frac{d}{dz}$, where z is the complex coordinate on $\mathbb{C}$. Prove that $\operatorname{Lie}_\rho |z|^2 = |z|^2$.

b. Prove that $\operatorname{Lie}_\rho(\log |z|) = \mathsf{const}$.

Exercise 4.4. Let $z_1, ..., z_{n+1}$ be the complex coordinates on $\mathbb{C}^{n+1}$.

a. Prove that the vector fields $r := \sum_{i=1}^{n+1} z_i \frac{d}{dz_i}$ and $\bar{r} := \sum_{i=1}^{n+1} \bar{z}_i \frac{d}{d\bar{z}_i}$ are $\mathbb{C}^*$-invariant.

b. (!) Let $l(z_1, ..., z_n) = \sum |z_i|^2$. Prove that $\operatorname{Lie}_r(\log l) = 0$.

c. (!) Prove that $i_r(d \log l) = i_{\bar{r}}(d \log l) = 0$.

Hint. Use the previous exercise.

Definition 4.1. Let $E \subset TM$ be a foliation, $M_0 \subset M$ an open subset and $\pi : M_0 \longrightarrow X$ its leaf space, that is, a submersion with connected fibers which satisfies $d\pi = E$. A differential form η on M is called **basic** with respect to E if for any open subset $M_0 \subset M$ and the leaf space $\pi : M_0 \longrightarrow X$, there exists a form η_0 on M_0 such that $\eta\Big|_{M_0} = \pi^*\eta_0$.

Exercise 4.5. Prove that a form η is basic with respect to $E \subset TM$ if and only if $i_X\eta = 0$ and $\operatorname{Lie}_x \eta = 0$ for any vector field $X \in E$.

Exercise 4.6. Let η be a closed form. Prove that η is basic with respect to $E \subset TM$ if and only if $i_X\eta = 0$.

Exercise 4.7. Consider the tautological fibration $\mathbb{C}^{n+1}\backslash 0 \xrightarrow{\pi} \mathbb{C}P^n$.

a. Prove that there exists a form $\omega \in \Lambda^{1,1}(\mathbb{C}P^n)$ such that $dd^c \log l = \pi^*(\omega)$.

b. Prove that this form is $U(n)$-invariant and has at least one positive eigenvalue.

c. (!) Prove that ω is a Kähler form.

Definition 4.2. A Fubini-Study form on $\mathbb{C}P^n$ is an $U(n)$-invariant Hermitian form.

Exercise 4.8 (!). Let $\mathbb{C}^n \subset \mathbb{C}P^n$ be an affine chart with affine coordinates $z_1, ..., z_n$. Prove that the Fubini-Study form in this chart is given by

$$\omega = \frac{\sum_{i=1}^n dz_i \wedge d\bar{z}_i}{1 + \sum_{i=1}^n |z_i|^2} - \frac{\sum_{i=1}^n \bar{z}_i dz_i}{1 + \sum_{i=1}^n |z_i|^2} \wedge \frac{\sum_{i=1}^n z_i d\bar{z}_i}{1 + \sum_{i=1}^n |z_i|^2}$$