Complex surfaces, home assignment 5: Currents and Hahn-Banach theorem

Rules: This is a class assignment for this week, for discussion in class Wednesdays and Fridays after the lecture.

Remark 5.1. The next 3 exercises are solved using the same Hahn-Banach trick as in lectures: either a cone $A \subset V$ intersects a closed subspace $W \subset V$, or you have a functional $\lambda \in V^*$ vanishing on W and positive on A. This argument, as applied to complex geometry, is due to Sullivan and Harvey-Lawson.

Exercise 5.1. A complex manifold is called **Hermitian symplectic** if it admits a symplectic form with (1,1)-part Hermitian. Prove that a compact complex n-manifold is either Hermitian symplectic, or admits an exact, positive $(n-1, n-1)$-current.

Exercise 5.2. A compact complex n-manifold M is called **balanced** if it admits a Hermitian (1,1)-form ω such that ω^{n-1} is closed. Prove that either M is balanced or it admits an exact 2-current with (1,1)-part positive and non-zero.

Exercise 5.3. A compact complex n-manifold M is called **exact balanced** if it admits a Hermitian (1,1)-form ω such that ω^{n-1} is exact. Prove that either M is exact balanced or it admits a closed 2-current with (1,1)-part positive and non-zero.

Definition 5.1. Let M be a compact complex n-manifold. Its **The Gauduchon cone** is the set of all Aeppli classes of ω^{n-1}, where ω is a Gauduchon metric.

Exercise 5.4. Prove that the Gauduchon cone is open in $H^{n-1,n-1}_{AE}(M, \mathbb{R})$.

Exercise 5.5. Prove that the Gauduchon cone contains 0 if and only if M does not admit non-zero positive closed (1,1)-currents.

Exercise 5.6. Let M be a compact homogeneous manifold admitting a Kähler current. Prove that it admits a Kähler form.