Complex surfaces, 2025, exam

Rules: Every student receives from me a list of 10 exercises (chosen randomly), and has to solve as many of them as you can before April 1, 2025. Please write down the solution and bring it to exam for me to see. Marks: C when $20 \leqslant N < 30$, B when $30 \leqslant N < 50$, A when $50 \leqslant N \leqslant 70$, A+ when $N > 70$; here N is the sum of points.

1 Elliptic operators

Exercise 1.1 (10 points). Let M be a Riemannian manifold, and D an elliptic operator of second order. Prove that there exists a Riemannian metric g, a vector field v and a function $c \in C^\infty M$ such that $D(f) = \pm\Delta(f) + \operatorname{Lie}_v(f) + c_1 f$.

Exercise 1.2 (20 points). Consider the standard action of $SO(n+1)$ on S^n, and let D be an $SO(n+1)$-invariant second order differential operator. Prove that $D(f) = af + b\Delta(f)$, where Δ is the usual Laplacian associated with the standard metric, and $a, b \in \mathbb{R}$.

Exercise 1.3 (10 points). Let (M, g) be a compact Riemannian manifold, and Δ its Laplacian. Prove that all solutions of $\Delta(\Delta(f)) = 0$, $f \in C^\infty M$ are constant.

Exercise 1.4 (10 points). Let f be a smooth function on a compact Riemannian manifold, such that $\Delta(f) = \lambda f$, where $\lambda \in C^\infty M$ is a negative function. Prove that $f = 0$.

Exercise 1.5 (20 points). Let $D(f) := \Delta(\Delta(f)) + f$ be a differential operator on compact Riemannian manifold, where Δ is Laplacian. Prove that D is surjective.

2 Positive currents and plurisubharmonic functions

Exercise 2.1 (20 points). Let M be a compact homogeneous manifold admitting a Kähler current. Prove that it admits a Kähler form.

Exercise 2.2 (10 points). Let V be the space of all currents x which satisfy $\langle x, \tau_i \rangle \geqslant 0$ for a linearly independent collection of forms $\tau_1, ..., \tau_k$. Prove that V contains a current which is not positive.

Exercise 2.3 (10 points). Let η be a positive, closed current on a complex surface M, smooth outside of $x \in M$. Assume that $\eta = dd^c f$ in a neighbourhood $U \ni x$, and $\lim_{y \to x} f(y) = -\infty$. Prove that η is cohomologous to a smooth closed form η_1 such that $\int_M \eta_1 \wedge \eta_1 > 0$.

Exercise 2.4 (10 points). Let ω be a symplectic form on a compact complex n-manifold M Assume that its (1,1)-part is Hermitian. Let η be a positive exact (1,1)-current on M. Prove that $\eta = 0$.

Exercise 2.5 (10 points). Let f be a function on $\mathbb{C}^2\backslash 0$, such that $dd^c f = 0$. Prove that f can be smoothly extended to a function on $\mathbb{C}^2$ with $dd^c f = 0$.

Exercise 2.6 (20 points). Let η be a positive, closed $(1,1)$-current on a complex surface. Assume that $\eta = 0$ outside of a finite set. Prove that $\eta = 0$.

Hint. Use the previous exercise.

Exercise 2.7 (20 points). Let f be a smooth convex function on an open ball B, such that $dd^c f > 0$ outside of 0. Prove that there exists a smooth function f_1 which is strictly plurisubharmonic and equal to f outside of a ball $B_\varepsilon(0)$ of radius ε.

3 Hermitian forms and positivity

Exercise 3.1 (20 points). Let (M, ω) be a compact complex Hermitian 3-manifold. Assume that $dd^c\omega = 0$. Prove that any holomorphic 1-form on M is closed.

Exercise 3.2 (10 points). Let G be a compact group acting on a compact complex manifold M by holomorphic diffeomorphisms. Prove that there exists a G-invariant Gauduchon metric.

Exercise 3.3 (10 points). Let G be a Lie group transitively acting on a compact complex manifold M by holomorphic diffeomorphisms, and ω a G-invariant Hermitian form. Prove that ω is Gauduchon.

Exercise 3.4 (20 points). Let ω be a Hermitian form on a compact complex manifold M, $\dim_{\mathbb{C}} M > 2$. Assume that $d\omega = \omega \wedge \theta$, for some 1-form θ and $dd^c\omega = 0$. Prove that $d\omega = 0$.

Exercise 3.5 (20 points). Let ω be a Hermitian form on a compact complex n-manifold M. Assume that $d(\omega^{n-1}) = 0$, and $dd^c\omega = 0$. Prove that $d\omega = 0$.

4 Complex surfaces

Exercise 4.1 (10 points). Let $C \subset M$ be connected complex curve on a a complex surface, and $C_1, ..., C_k$, $k > 1$ its irreducible components. Assume that C is homologous to 0. Prove that each of the curves C_i has negative self-intersection.

Exercise 4.2 (10 points). Let M be a compact complex surface, and ω a Gauduchon form. Prove that its class in $H^{1,1}_{AE}(M)$ is non-zero.

Exercise 4.3 (20 points). Let M be a complex surface with $b_2 = 0$. Prove that every line bundle on M admits a flat connection, compatible with a holomorphic structure.

Exercise 4.4 (10 points). Let ω be a symplectic form on a complex surface M. Assume that its (1,1)-part is Hermitian. Prove that defect of M vanishes (and hence M is Kähler).

Definition 4.1. Let E be an elliptic curve, and L a holomorphic line bundle on E, $\deg L \neq 0$. Consider the space $\mathrm{Tot}^\circ(L)$ of all non-zero vectors in the total space of L, and let λ be a complex number ith $|\lambda| > 0$. Consider the $\mathbb{Z}$ action on $\mathrm{Tot}^\circ(L)$ generated by $v \mapsto \lambda v$. The quotient $\mathrm{Tot}^\circ(L)/\mathbb{Z}$ is called **Kodaira surface**. It is projected to E with the fiber $\mathbb{C}^*/\langle\lambda\rangle$, also isomorphic to an elliptic curve.

Exercise 4.5 (10 points). Let S be a complex curve on a Kodaira surface. Prove that $\pi(S) \subset E$ is 0-dimensional.

Exercise 4.6 (10 points). Prove that the Kodaira surface admits a non-degenerate, closed holomorphic 2-form.

5 Hahn-Banach theorem, Bott-Chern and Aeppli cohomology

Exercise 5.1 (10 points). Let M be a manifold such that the natural map $H^{1,1}_{BC}(M) \longrightarrow H^2(M)$ is injective. Prove that any class in $H^1(M)$ can be represented by a form which belongs to $\ker d \cap \ker d^c$.

Exercise 5.2 (10 points). A complex manifold is called **Hermitian symplectic** if it admits a symplectic form with (1,1)-part Hermitian. Prove that a compact complex n-manifold is either Hermitian symplectic, or admits an exact, positive $(n-1, n-1)$-current.

Exercise 5.3 (10 points). A compact complex n-manifold M is called **balanced** if it admits a Hermitian (1,1)-form ω such that ω^{n-1} is closed. Prove that either M is balanced or it admits an exact 2-current with (1,1)-part positive and non-zero.

Exercise 5.4 (10 points). A compact complex n-manifold M is called **exact balanced** if it admits a Hermitian (1,1)-form ω such that ω^{n-1} is exact. Prove that either M is exact balanced or it admits a closed 2-current with (1,1)-part positive and non-zero.

Exercise 5.5 (10 points). Let M be a compact complex manifold not admitting a dd^c-closed Hermitian metric. Prove that M admits a dd^c-exact positive $(n-1, n-1)$-current.

Exercise 5.6 (10 points). Prove that a compact complex manifold M admits a Gauduchon form which is cohomologous to 0 in $H^{n-1,n-1}_{AE}(M)$ if and only if M does not admit non-zero positive closed (1,1)-currents.