Complex surfaces

lecture 1: Kodaira-Enriques classification for non-algebraic surfaces

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Kodaira dimension

REMARK: This is an introductory lecture. Kodaira classification **will not be proven in this course** (see [BHPV] for its proof); we also assume existence of minimal models.

DEFINITION: A complex surface is a compact complex manifold M of complex dimension 2. Let M be a compact complex manifold, and K_M its canonical bundle. The **canonical ring** $\bigoplus_{i=0}^\infty H^0(K^i)$ is finitely generated for for all projective varieties (Birkar, Cascini, Hacon, McKernan), for complex surfaces (Kodaira). Conjecturally, it is **always finitely generated.** Let $a \in \mathbb{Z}^{>0}$. Consider the function $P_a(N) = H^0(K^{aN})$. If the canonical ring is finitely generated, the function $N \mapsto P_a(N)$ **is polynomial** for a which divides all degrees of its generators **(prove this).** The degree $\kappa(M)$ of this polynomial is called **the Kodaira dimension** of M. If $H^0(K^i) = 0$ for all $i > 0$, we set $\kappa(M) = -\infty$.

REMARK: Projective surfaces were classified by Enriques; Kodaira explained his classification, using the Kodaira dimension, and extended it to complex surfaces.

Today, **I will formulate Kodaira classification theorem for non-projective complex surfaces.**

Nilmanifolds

DEFINITION: Let M be a smooth manifold equipped with a transitive action of a nilpotent Lie group. Then M is called **a nilmanifold**.

REMARK: All nilmanifolds are obtained as quotient spaces, $M = G/H$.

THEOREM: (Malčev)
Let $\mathfrak{g}$ be a nilpotent Lie algebra defined over $\mathbb{Q}$, and G its Lie group. **Then G contains a discrete subgroup Γ such that G/Γ is compact,** and $\Gamma = e^{\Gamma_{\mathfrak{g}}}$, where $\Gamma_{\mathfrak{g}}$ is a lattice subalgebra in $\mathfrak{g}$. Moreover, $\mathfrak{g} \cong \Gamma_{\mathfrak{g}} \otimes_{\mathbb{Q}} \mathbb{R}$. Finally, **all nilmanifolds are obtained this way.**

REMARK: Topologically, **all simply connected nilpotent Lie groups are diffeomorphic to $\mathbb{R}^n$,** and all nilmanifolds are **iterated circle fibrations**.

Complex structures

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I : TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

REMARK: It is sufficient to check the condition $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$ on any set of generators of $T^{1,0}M$. In particular, if (M,I) is homogeneous (equipped with a transitive Lie group action preserving I) **it suffices to check it on invariant vector fields**.

Complex nilmanifolds

DEFINITION: An integrable complex structure on a real Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{g}^{1,0} \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ such that $\mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$

REMARK: Any such decomposition defines a complex structure I on $\mathfrak{g}$ by $I\big|_{\mathfrak{g}^{1,0}} = \sqrt{-1}$ and $I\big|_{\mathfrak{g}^{0,1}} = -\sqrt{-1}$. Integrability of complex structure is given by $[T^{1,0}G, T^{1,0}G] \subset T^{1,0}G$, which is equivalent to $[\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}] \subset \mathfrak{g}^{1,0}$.

REMARK: Right-invariant complex structures on a connected real Lie group **are in 1 to 1 correspondence with integrable complex structures** on its Lie algebra.

DEFINITION: A **complex nilmanifold** is a nilmanifold $M = G/\Gamma$ equipped with a complex structure, in such a way that G has a right-invariant complex structure, and the projection $G \longrightarrow M$ is holomorphic.

REMARK: The same way one defines **complex solvmanifolds** $M = G/\Gamma$, where G is a solvable Lie group, Γ a cocompact lattice, and $\mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ a decomposition of its Lie algebra. **Indeed, this construction would work for any Lie group,** giving left-invariant complex structures on G and its left quotients by any discrete subgroup.

Kodaira surface as a complex nilmanifold

Let $G := \mathbb{R} \times G_0$, where G_0 is the 3-dimensional real Lie group of upper triangular matrices 3x3. This group contains many cocompact lattices, for example $\Gamma := \mathbb{Z} \times \Gamma_0$, where Γ_0 of integer matrices. The corresponding Lie algebra $\mathfrak{g}$. **is generated by x, y, z, t with the only non-zero commutator $[x, y] = z$.**

DEFINITION: (Primary) Kodaira surface can be defined as $M := G/\Gamma$ with the complex structure defined by the subalgebra $\mathfrak{g}^{1,0} := \langle x + \sqrt{-1}\, y, z + \sqrt{-1}\, t\rangle$, which is actually abelian.

Kodaira surfaces

DEFINITION: **Primary Kodaira surface** (also called **Kodaira-Thurston surface**) is a complex surface M with $b_1(M) = 3$ admitting locally trivial holomorphic fibration with the fiber elliptic curve over a base elliptic curve. **Secondary Kodaira surface** is a quotient of a primary Kodaira surface by a free action of a finite group, with $b_1(M) = 1$.

EXAMPLE: Let $G := \mathbb{R} \times G_0$ be as above, and consider the fibration $G/\Gamma \longrightarrow \frac{G}{Z}\Big/\frac{\Gamma}{\mathbb{Z}^2}$, where Z is the center of G, and $\mathbb{Z}^2$ in the second term is its intersection with the lattice. On the level of Lie algebras, this projection corresponds to the map $\mathfrak{g} = \langle x, y, z, t\rangle \longrightarrow \langle x, y\rangle$.

REMARK: From the construction of the complex structure on M, **it is apparent that this projection is holomorphic**; its fibers and its base **is 2-dimensional nilmanifolds, that is, elliptic curves. This implies that G/Γ is a primary Kodaira surface.**

REMARK: For a complete classification of 2-dimensional complex nilmanifolds and solvmanifolds, see *Keizo Hasegawa, Complex and Kahler structures on Compact Solvmanifolds, J. Symplectic Geom. Volume 3, Number 4 (2005), 749-767,* `https://arxiv.org/abs/0804.4223`.

Minimal models

REMARK: It is not hard to see that **bimeromorphic complex surfaces have the same Kodaira dimension.**

DEFINITION: Let M be a complex surface. It is called **minimal** if it does not contain a smooth rational curve with self-intersection -1.

THEOREM: (Castelnuovo) For any complex surface M, **there exists a minimal surface M_1 and a holomorphic, bimeromorphic map $M \longrightarrow M_1$.**

DEFINITION: In this situation, M_1 is called **a minimal model** for M.

EXERCISE: Observe that **a minimal model is not unique;** discuss examples.

Neron-Severi lattice for non-algebraic complex surfaces

DEFINITION: Let M be a compact Kähler manifold. **The Neron-Severi lattice** is $NS(M) := H^{1,1}(M) \cap H^2(M, \mathbb{Z})$.

THEOREM: (Kodaira)
A complex surface **is projective if any of these equivalent statements holds.**
(i) The field of meromorphic functions on M has trancendental dimension 2.
(ii) M admits a holomorphic line bundle L with $c_1(L)^2 > 0$.
(iii) The Neron-Severi lattice of M contains a class with positive self-intersection.

REMARK: This statement is not hard to deduce from Kodaira vanishing theorem and Nakai-Moishezon theorem.

Remark 1: From this theorem it follows immediately that for any non-algebraic surface, the intersection form on $NS(M)$ is non-positive definite. In particular, **for any $a, b \in NS(M)$ with $a^2 = 0$, we have $ab = 0$.** Indeed, if $ab > 0$, the square $(na + b)^2$ is positive for $n \gg 0$.

Class VII surfaces

DEFINITION: Class VII surface (also called Kodaira class VII surface) is a complex surface with $\kappa(M) = \infty$ and first Betti bumber $b_1(M) = 1$. Minimal class VII surfaces are called **class VII$_0$ surfaces.**

REMARK: Kodaira defined the "Class VII" in another, non-equivalent way. The current "Class VII" is Kodaira's "class 7" from his version of Kodaira-Enriques classification, published 1966. The term "class VII" with its current meaning is due to Barth, Peters, Van de Ven.

Kodaira classification for non-algebraic surfaces

THEOREM: Let M be a minimal complex surface which is not projective. **Then M belongs to one of the following mutually exclusive classes.**

[$\kappa(M) = 1$] M is equipped with a folomorphic fibration $\pi: M \longrightarrow S$ over a complex curve S of genus > 1. All fibers of π are elliptic curves, and general fibers are isomorphic.

[$\kappa(M) = 0$] M is a K3 surface, torus, primary or secondary Kodaira surface.

[$\kappa(M) = -\infty$] M is a class VII_0 surface.

REMARK: If $\kappa(M) = 1$, the surface M admits infinitely many curves in the same homology class $[C]$. Since the self-intersection form on $NS(M)$ is non-positive definite, we have $[C]^2 = 0$. This means that these curves do not intersect, and **M is holomorphically fibered over a curve.** The canonical class of the general fiber C of this fibration is obtained by adjunction formula, $K_C = K_M|_C$, because the normal bundle is trivial. Since $[C]^2 = 0$, $x \cdot [C] = 0$ for any $x \in NS(M)$ (Remark 1), which implies that $K_C = K_M|_C$ has degree 0. **Therefore, C is an elliptic curve, and M is an elliptic surface.** This proves that **any non-algebraic surface with $\kappa(M) = 1$ is elliptic.**

REMARK: The same argument also implies that **the elliptic fibration on M is unique,** for any non-projective elliptic surface M.