Complex surfaces

Lecture 2: Hopf manifolds and algebraic cones

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Kodaira dimension (reminder)

DEFINITION: A complex surface is a compact complex manifold M of complex dimension 2.

DEFINITION: A complex surface is a compact complex manifold M of complex dimension 2. Let M be a compact complex manifold, and K_M its canonical bundle. The **canonical ring** $\bigoplus_{i=0}^{\infty} H^0(K^i)$ is finitely generated for for all projective varieties (Birkar, Cascini, Hacon, McKernan), for complex surfaces (Kodaira). Conjecturally, it is **always finitely generated.** Let $a \in \mathbb{Z}^{>0}$. Consider the function $P_a(N) = H^0(K^{aN})$. If the canonical ring is finitely generated, the function $N \mapsto P_a(N)$ **is polynomial** for a which divides all degrees of its generators **(prove this).** The degree $\kappa(M)$ of this polynomial is called **the Kodaira dimension** of M. If $H^0(K^i) = 0$ for all $i > 0$, we set $\kappa(M) = -\infty$.

Holomorphic contractions and Hopf manifolds

DEFINITION: A **holomorphic contraction** of a manifold M with center in $x \in M$ is a map $\gamma: M \longrightarrow M$, $\gamma(x) = x$, such that for each compact subset $K \subset M$ and any open neighbourhood $U \ni x$, there exists $N > 0$ such that $\gamma^n(K) \subset U$ for all $n > N$.

DEFINITION: Let $\gamma \in \operatorname{Aut}(\mathbb{C}^n)$ be an invertible contraction centered in 0, and $H := \frac{\mathbb{C}^n \backslash 0}{\langle \gamma \rangle}$ the corresponding $\mathbb{Z}$-quiotient. Then H is a complex manifold, called **a Hopf manifold**. A Hopf manifold H is called **a linear Hopf manifold** if the contraction γ is linear, and **a classical Hopf manifold** if $\gamma = \lambda \operatorname{Id}$.

PROPOSITION: A Hopf manifold **is diffeomorphic to $S^1 \times S^{2n-1}$**.
Proof. Step 1: If M is a classical Hopf manifold, this is clear (think of polar system of coordinates). If M is linear, we deform γ it to $\lambda \operatorname{Id}$, obtaining **a smooth family of compact manifolds with one of the fibers a classical Hopf manifold;** Ehresmann theorem implies that all fibers are diffeomorphic.

Step 2: Finally, if γ is an arbitrary contraction, we approximate γ by a linear contraction A (around 0, such approximation always exists, because γ is smooth), and consider a smooth family of Hopf manifolds defined by the contraction $tA + (1-t)\gamma$, with $t \in [0,1]$. Then we use Ehresmann theorem again to show that all fibers of this smooth family are diffeomorphic. ■

Class VII surfaces (reminder)

DEFINITION: Class VII surface (also called Kodaira class VII surface) is a complex surface with $\kappa(M) = \infty$ and first Betti bumber $b_1(M) = 1$. Minimal class VII surfaces are called **class $\mathbf{VII_0}$ surfaces.**

REMARK: Kodaira defined the "Class VII" in another, non-equivalent way. The current "Class VII" is Kodaira's "class 7" from his version of Kodaira-Enriques classification, published 1966. The term "class VII" with its current meaning is due to Barth, Peters, Van de Ven.

Hopf surfaces are class VII

DEFINITION: A **primary Hopf surface** is Hopf manifold of dimension 2. A **secondary Hopf surface** is a quotient of a primary Hopf surface H by a finite group freely and holomorphically acting on H.

CLAIM: Hopf surfaces are class VII$_0$.

Proof. Step 1: The Kodaira dimension is stable under finite quotients **(prove it).** It remains to show only that $H := \mathbb{C}^2 \backslash 0 / \langle\gamma\rangle$ is class VII$_0$.

Step 2: Since $\pi_1(\mathbb{C}P^1) = 0$, covering homotopy implies that any map $\varphi : \mathbb{C}P^1 \longrightarrow H$ is lifted to $\mathbb{C}^2\backslash 0$; since $\mathbb{C}^2\backslash 0$ does not contain compact curves, the Hopf surface does not contain rational curves. **This implies that H is minimal.**

Hopf surfaces are class VII (2)

Step 3: Let M be a complex n manifold. For any holomorphic n-form $\alpha \in \Omega^n(H)$, the form $\alpha \wedge \overline{\alpha}$ is a volume form, which is positive outside of the zero divisor of α. For any holomorphic section η of $\Omega^n(H)^{\otimes N}$, one can locally (outside of its zero set) define its N-th root α; then $\alpha \wedge \overline{\alpha}$ is a volume form. It is independent from the choice of the N-th root **(check this).** This volume form is strictly positive outside of zero set of η. One can consider $\eta \wedge \overline{\eta}$ as a section of the real bundle $\Lambda^{2n}(M)^{\otimes N}$, called **the bundle of plurivolume forms**. The corresponding volume form is denoted $\sqrt[N]{\eta \wedge \overline{\eta}}$

Step 4: It remains to show only that $\kappa(M) = 0$. We argue by contradiction. Let $\eta \in \Omega^2(H)^{\otimes N}$ be a non-zero section of the pluricanonical bundle $K_H^{\otimes N}$. The pullback of η is a holomorphic pluricanonical form $\tilde{\eta} \in \Omega^2(\mathbb{C}^2 \backslash 0)^{\otimes N}$, which is invariant under the contraction γ. By Riemann (or Hartogs) extension theorem, $\tilde{\eta}$ can be extended to a holomorphic pluricanonical form on $\mathbb{C}^2$. Let B be an open subset with compact closure such that $\gamma(B) \subset B$; then

$$\int_B \sqrt[N]{\tilde{\eta} \wedge \overline{\tilde{\eta}}} = \int_{\gamma(B)} \sqrt[N]{\tilde{\eta} \wedge \overline{\tilde{\eta}}}$$

by γ-invariance of $\tilde{\eta}$, and **this is impossible, because $\gamma(B)$ is strictly smaller than B, and the volume form $\sqrt[N]{\tilde{\eta} \wedge \overline{\tilde{\eta}}}$ is strictly positive outside of its zeros set.** ■

Algebraic cones

DEFINITION: Let P be a projective orbifold, and L an ample line bundle on P. Assume that the total space $\mathrm{Tot}^\circ(L)$ of all non-zero vectors in L is smooth. **An open algebraic cone** is $\mathrm{Tot}^\circ(L)$.

EXAMPLE: Let $P \subset \mathbb{C}P^n$, and $L = \mathcal{O}(1)|_P$. Then **the open algebraic cone** $\mathrm{Tot}^\circ(L)$ **can be identified with the set** $\pi^{-1}(P)$ of all $v \in \mathbb{C}^{n+1} \backslash 0$ projected to P under the standard map $\pi: \mathbb{C}^{n+1} \backslash 0 \to \mathbb{C}P^n$. **The closed algebraic cone** is its closure in $\mathbb{C}^{n+1}$. It is an affine subvariety in $\mathbb{C}^{n+1}$ given by the same collection of homogeneous equations as P. Its **origin** is zero.

REMARK: Clearly, an algebraic cone **is equipped with an invertible holomorphic contraction to its origin.**

Vaisman manifolds defined in terms of algebraic cones

DEFINITION: An automorphism $A: P \longrightarrow P$ is **L-linearizable** L admits an A-equivariant structure, in other words, if A can be lifted to an automorphism of the cone $\operatorname{Tot}^\circ(L)$ which is linear on fibers.

DEFINITION: Fix a Hermitian metric on L, and let $A: \operatorname{Tot}^\circ(L) \longrightarrow \operatorname{Tot}^\circ(L)$ be an automorphism which is linear on fibers and satisfies $|A(v)| = \lambda|v|$ for some number $\lambda < 1$. Then A acts on the closed cone as a holomorphic contraction, and the quotient space $\operatorname{Tot}^\circ(L)/\langle A\rangle$ is Hausdorff for the same reason as it is Hausdorff for a Hopf manifold (an action of an invertible contraction is always totally discontinuous outside of the origin; **prove this as an exercise).** The quotient manifold $\operatorname{Tot}^\circ(L)/\langle A\rangle$ is called **a Vaisman manifold**.

REMARK: This is not the standard definition of a Vaisman manifold; we will discuss several standard definitions in lecture 4. We will also prove that **Vaisman manifolds are never Kähler**.

Examples of Vaisman surfaces

EXAMPLE: The classical Hopf manifold **is Vaisman.** Indeed, $\mathbb{C}^2\backslash 0$ is identified with the total space $\operatorname{Tot}^\circ(\mathcal{O}(1))$ of $\mathcal{O}(1)$ on $\mathbb{C}P^1$, and $v \mapsto \lambda v$ is a contraction which acts linearly on the fibers.

EXAMPLE: The following result is non-trivial (it is due to Gauduchon-Ornea, Belgun and others). Let $H := \frac{\mathbb{C}^2\backslash 0}{\langle A\rangle}$ be a linear Hopf surface. Then H is Vaisman if and only if A is semisimple (that is, diagonalizable). In particular, **H is not Vaisman if A is a Jordan cell.**

EXAMPLE: Let L be an ample bundle over a compact orbi-curve S, and $v \stackrel{A}{\mapsto} \lambda v$ the contraction of $\operatorname{Tot}^\circ(L)$, where $|\lambda| < 1$. The corresponding quotient manifold $\operatorname{Tot}^\circ(L)/\langle A\rangle$ **is a Vaisman manifold, which is elliptically fibered over S.**

THEOREM: (Ornea-Vuletescu-V.) All non-Kähler elliptic surfaces are Vaisman and obtained from this construction.

Kodaira surfaces are Vaisman

EXAMPLE: Let L be an ample bundle over an elliptic curve S, and $M := \operatorname{Tot}^\circ(L)/\langle A\rangle$ obtained as above. Consider the exact sequence

$$0 \longrightarrow H^1(S) \xrightarrow{\pi^*} H^1(M) \xrightarrow{\tau^*} H^1(F) \xrightarrow{d_2} H^2(S) \xrightarrow{\pi^*} H^2(M) \longrightarrow H^2(F) \longrightarrow 0$$

where F is the fiber (also an elliptic curve). This exact sequence is obtained from the Leray-Serre spectral sequence (see `http://verbit.ru/IMPA/K3-2024/assign-02-K3-2024.pdf`). Since d_2 is the first Chern class of L, it has rank 1, hence $b_2(M) = 3$. Therefore, M **is a Kodaira surface.** The same argument as proves Ornea-Vuletescu-V. theorem implies that **all primary Kodaira surfaces are obtained this way.**

REMARK: A smooth finite quotient of a Vaisman manifold is Vaisman; this includes the secondary Hopf and secondary Kodaira surfaces.