## Complex surfaces, lecture 3:

# Chern connection

IMPA, sala 236

Misha Verbitsky, January 10, 2023, 17:00

http://verbit.ru/IMPA/Surfaces-2025/

## Connections

**DEFINITION:** Recall that **a connection** on a bundle $B$ is an operator $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, where $f \longrightarrow df$ is de Rham differential. When $X$ is a vector field, we denote by $\nabla_X(b) \in B$ the term $\langle \nabla(b), X\rangle$.

**REMARK: A connection $\nabla$ on $B$ gives a connection $B^* \xrightarrow{\nabla^*} \Lambda^1 M \otimes B^*$ on the dual bundle,** by the formula

$$d(\langle b, \beta\rangle) = \langle \nabla b, \beta\rangle + \langle b, \nabla^* \beta\rangle$$

These connections are usually denoted **by the same letter $\nabla$.**

**REMARK:** For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on $B$ defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

## Parallel transport along the connection

**THEOREM:** Let $B$ be a vector bundle with connection over $\mathbb{R}$. Then for each $x \in \mathbb{R}$ and each vector $b_x \in B|_x$ **there exists a unique section** $b \in B$ **such that** $\nabla b = 0$**,** $b|_x = b_x$**.**

**Proof:** This is existence and uniqueness of solutions of an ODE $\frac{db}{dt} + A(b) = 0$. ■

**DEFINITION:** Let $\gamma : [0,1] \longrightarrow M$ be a smooth path in $M$ connecting $x$ and $y$, and $(B, \nabla)$ a vector bundle with connection. Restricting $(B, \nabla)$ to $\gamma([0,1])$, we obtain a bundle with connection on an interval. Solve an equation $\nabla(b) = 0$ for $b \in B\Big|_{\gamma([0,1])}$ and initial condition $b|_x = b_x$. This process is called **parallel transport** along the path via the connection. The vector $b_y := b|_y$ is called **vector obtained by parallel transport of** $b_x$ **along** $\gamma$. **Holonomy group** of $\gamma$ is the group of endomorphisms of the fiber $B_x$ obtained from parallel transports along all paths starting and ending in $x \in M$

## Curvature

Let $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a vector bundle $B$. **We extend $\nabla$ to an operator**

$$B \xrightarrow{\nabla} \Lambda^1(M) \otimes B \xrightarrow{\nabla} \Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B \xrightarrow{\nabla} ...$$

**using the Leibnitz identity** $\nabla(\eta \otimes b) = d\eta \otimes b + (-1)^{\tilde\eta} \eta \wedge \nabla b$**.**

**REMARK:** This operation is well defined, because

$$\nabla(\eta \otimes fb) = d\eta \otimes fb + (-1)^{\tilde\eta} \eta \wedge \nabla(fb) = \\ d\eta \otimes fb + (-1)^{\tilde\eta} \eta \wedge df \otimes b + f\eta \wedge \nabla b = d(f\eta) \otimes b + f\eta \wedge \nabla b = \nabla(f\eta \otimes b)$$

**REMARK:** Sometimes $\Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B$ is denoted $d_\nabla$.

**DEFINITION:** The operator $\nabla^2: B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of $\nabla$.

**REMARK: The algebra of differential forms with coefficients in** $\operatorname{End} B$ **acts on $\Lambda^* M \otimes B$** via $\eta \otimes a(\eta' \otimes b) = \eta \wedge \eta' \otimes a(b)$, where $a \in \operatorname{End}(B)$, $\eta, \eta' \in \Lambda^* M$, and $b \in B$. **This is the formula expressing the action of $\nabla^2$ on $\Lambda^* M \otimes B$.**

## Bianchi identity

**REMARK:** The algebra of $\operatorname{End}(B)$-valued forms naturally acts on $\Lambda^* M \otimes B$. The curvature satisfies $\nabla^2(fb) = d^2 fb + df \wedge \nabla b - df \wedge \nabla b + f\nabla^2 b = f\nabla^2 b$, hence it is $C^\infty M$-linear. **We consider it as an $\operatorname{End}(B)$-valued 2-form on $M$.**

**REMARK: (Bianchi identity)**
Clearly, $[\nabla, \nabla^2] = 0$. This gives **the Bianchi identity:** $d_\nabla(\Theta_B) = 0$, where $\Theta$ is considered as a section of $\Lambda^2(M) \otimes \operatorname{End}(B)$, and $d_\nabla : \Lambda^2(M) \otimes \operatorname{End}(B) \longrightarrow \Lambda^3(M) \otimes \operatorname{End}(B)$ the operator defined above.

**REMARK:** This implies that $d(\operatorname{Tr}_B(\Theta_B^n)) = 0$, hence $\operatorname{Tr}_B(\Theta_B^n)$ **is a closed differential form on $M$**. This form is used to define the Chern classes and Pontryagin classes ("Chern-Weil formula").

## Riemann-Hilbert correspondence

**DEFINITION:** A connection is **flat** if its curvature vanishes.

**THEOREM:** Let $M$ be a simply connected manifold, and $(B, \nabla)$ a bundle with connection. Then **$\nabla$ is flat if and only if the holonomy of $\nabla$ is trivial.**

**THEOREM:** Let $M$ be a connected manifold, $\mathcal{C}_1$ the category of representations of $\pi_1(M)$, and $\mathcal{C}_2$ the category of locally constant sheaves. **Then the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are naturally equivalent.**

**THEOREM:** The categories $\mathcal{C}_1$ and $\mathcal{C}_2$ **are naturally equivalent to the category of vector bundles on $M$ equipped with flat connection.**

## Holomorphic bundles

**DEFINITION: Holomorphic vector bundle** on a complex manifold $M$ is a locally trivial sheaf of $\mathcal{O}_M$-modules.

**DEFINITION: The total space** $\operatorname{Tot}(B)$ of a holomorphic bundle $B$ over $M$ is the space of all pairs $\{x \in M, b \in B_x/\mathfrak{m}_x B\}$, where $B_x$ is the stalk of $B$ in $x \in M$ and $\mathfrak{m}_x$ the maximal ideal of $x$. We equip $\operatorname{Tot}(B)$ with the natural topology and holomorphic structure, in such a way that $\operatorname{Tot}(B)$ becomes a locally trivial holomorphic fibration with fiber $\mathbb{C}^r$, $r = \operatorname{rk} B$.

**REMARK: The set of holomorphic sections of a map $\operatorname{Tot}(B) \longrightarrow M$ is naturally identified with the set of sections of the sheaf $B$.**

**CLAIM:** Ler $B$ be a holomorphic bundle. Consider the sheaf $B_{C^\infty} := B \otimes_{\mathcal{O}_M} C^\infty M$. **Then $B_{C^\infty}$ is a locally trivial sheaf of $C^\infty M$-modules**.

**DEFINITION:** $B_{C^\infty}$ is called **smooth vector bundle underlying the holomorphic vector bundle $B$**.

**REMARK:** The natural map $\operatorname{Tot}(B) \longrightarrow \operatorname{Tot}(B_{C^\infty})$ is a diffeomorphism.

## $\overline{\partial}$-operator on vector bundles

**REMARK:** Let $M$ be a complex manifold. Then **the operator $\overline{\partial}: C^\infty M \longrightarrow \Lambda^{0,1}(M)$ is $\mathcal{O}_M$-linear.**

**DEFINITION:** Let $B$ be a holomorphic vector bundle on $M$. Consider an operator $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ mapping $b \otimes f$ to $b \otimes \overline{\partial} f$, where $b$ is a holomorphic section of $B$, and $f$ smooth. This operator is called **a holomorphic structure operator** on $B$. **It is well-defined because $\overline{\partial}$ is $\mathcal{O}_M$-linear**, and $B_{C^\infty} = B \otimes_{\mathcal{O}_M} C^\infty M$.

**REMARK:** The kernel of $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ **coincides with the image of $B$** under the natural sheaf embedding $B \hookrightarrow B_{C^\infty}$, with $b \longrightarrow b \otimes 1$.

**DEFINITION:** A **$\overline{\partial}$-operator** on a smooth complex vector bundle $V$ over a complex manifold is a differential operator $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V$ satisfying $\overline{\partial}(fb) = \overline{\partial}(f) \otimes b + f\overline{\partial}(b)$ for any $f \in C^\infty M, b \in V$.

**REMARK:** A $\overline{\partial}$-operator **can be extended to**

$$\overline{\partial}: \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V,$$

using the Leibnitz identity $\overline{\partial}(\eta \otimes b) = \overline{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \overline{\partial}(b)$, for all $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

## Holomorphic structure operator

**REMARK: For any holomorphic bundle, one has $\overline{\partial}^2 = 0$.** Indeed, a holomorphic bundle admits a local trivialization.

**THEOREM: (Koszul-Malgrange)** Let $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ be a $\overline{\partial}$-operator on a complex vector bundle, satisfying $\overline{\partial}^2 = 0$, where $\overline{\partial}$ is extended to

$$V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V \xrightarrow{\overline{\partial}} \Lambda^{0,2}(M) \otimes V \xrightarrow{\overline{\partial}} \Lambda^{0,3}(M) \otimes V \xrightarrow{\overline{\partial}} ...$$

as above. **Then $B := \ker \overline{\partial} \subset V$ is a holomorphic bundle of the same rank, and $V = B_{\mathbb{C}^\infty}$.**

**Proof:** The proof uses the same argument as used to prove the Newlander-Nirenberg theorem. ■

**DEFINITION: A holomorphic structure operator** on a bundle $V$ is a $\overline{\partial}$-operator $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ which satisfies $\overline{\partial}^2 = 0$.

**COROLLARY:** The category of holomorphic vector bundles **is equivalent to the category of complex vector bunldes equipped with a holomorphic structure operator.**

## Connections and holomorphic structures

**DEFINITION:** Let $V$ be a smooth complex vector bundle with connection $\nabla : V \longrightarrow \Lambda^1(M) \otimes V$ and holomorphic structure $\overline{\partial} : V \longrightarrow \Lambda^{0,1}(M) \otimes V$. Consider the Hodge type decomposition of $\nabla$, $\nabla = \nabla^{0,1} + \nabla^{1,0}$, where

$$\nabla^{0,1} : V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0} : V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that **the connection $\nabla$ is compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

**DEFINITION:** **A holomorphic Hermitian vector buncle** is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure.

**DEFINITION:** Let $(B, \nabla)$ be a bundle with connection, and $h$ a Hermitian form on $B$. Extending $\nabla$ to the tensor powers of $B$, we can apply it to $h$; the result can be obtained explicitly from the formula

$$(\nabla h)(X, Y) = d(h(X, Y)) - h(\nabla X, Y) - h(X, \nabla Y).$$

The connection $\nabla$ is called **unitary** if $\nabla(h) = 0$.

**DEFINITION:** **Chern connection** on a holomorphic Hermitian vector bundle is a unitary connection compatible with the holomorphic structure.

## Chern connection

**THEOREM:** Every holomorphic Hermitian vector bundle **admits a Chern connection, which is unique.**

**Proof. Step 1:** Given a complex vector bundle $B$, define **complex conjugate bundle** $\overline{B}$ as the same $\mathbb{R}$-bundle with complex conjugate $\mathbb{C}$-action. Then **a connection $\nabla$ on $B$ defines a connection $\overline{\nabla}$ on $\overline{B}$, with $\overline{\nabla}^{1,0} = \overline{\nabla^{0,1}}$ and $\overline{\nabla}^{0,1} = \overline{\nabla^{1,0}}$.**

**Step 2:** Define **$\nabla^{1,0}$-operator** on a complex vector bundle $B$ as a map $B \xrightarrow{\nabla^{1,0}} \Lambda^{1,0}(M) \otimes B$, satisfying $\Lambda^{1,0}(fb) = \partial(f) \otimes b + f\nabla^{1,0}(b)$ for any $f \in C^\infty M, b \in B$. **A $\overline{\partial}$-operator on $B$ defines an $\nabla^{1,0}$-operator on $\overline{B}$, and vice versa.**

**Step 3:** Hermitian form defines an isomorphism of complex vector bundles $B \xrightarrow{g} \overline{B}^*$. Holomorphic structure on $B$ defines a $\overline{\partial}$-operator on $\overline{B} = B^*$, which is the same as $\nabla^{1,0}$-operator $\nabla_g^{1,0}$ on $B$. **This gives a connection operator $\nabla := \overline{\partial} + \nabla_g^{1,0}$ on $B$,** which is Hermitian by construction. ■

**REMARK:** When people say about "curvature of a holomorphic Hermitian line bundle", **they speak about curvature of the Chern connection.**

## Curvature of a holomorphic line bundle

**REMARK:** If $L$ is a line bundle, $\operatorname{End} L$ is trivial, and **the curvature $\Theta_L$ of $L$ is a closed 2-form.**

**REMARK:** When speaking of a **"curvature of a holomorphic bundle"**, one usually means the curvature of a Chern connection.

**REMARK:** Let $B$ be a holomorphic Hermitian line bundle, and $b$ its non-degenerate holomorphic section. Denote by $\eta$ a (1,0)-form which satisfies $\nabla^{1,0} b = \eta \otimes b$. Then $d|b|^2 = \operatorname{Re} g(\nabla^{1,0} b, b) = \operatorname{Re} \eta |b|^2$. **This gives** $\nabla^{1,0} b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b$.

**REMARK:** Then $\Theta_B(b) = 2\bar\partial\partial \log |b| b$, **that is,** $\Theta_B = -2\partial\bar\partial \log |b|$.

**COROLLARY:** If $g' = e^{2f} g$ – two metrics on a holomorphic line bundle, $\Theta, \Theta'$ their curvatures, **one has** $\Theta' - \Theta = -2\partial\bar\partial f$

## Curvature of the Chern connection

**PROPOSITION: Curvaure $\Theta_B$ of a Chern connection on $B$ is a (1,1)-form: $\Theta_B \in \Lambda^{1,1}(M) \otimes \operatorname{End}(B)$.**

**Proof. Step 1:** Let $B$ be a Hermitan bundle. Consider the operator $\varphi \xrightarrow{\iota} -\varphi^*$ acting on $\operatorname{End}(B)$, where $\varphi \longrightarrow \varphi^*$ denotes the Hermitian conjugation. Since $\iota^2 = \operatorname{Id}$, and this is an anticomplex operator, it defines the real structure, and its fixed point set is $\mathfrak{u}_B$, the Lie algebra of anti-Hermitian matrices. **Step 2:** Since the Chern connection preserves the Hermitian structure $g$, one has $\nabla(g) = 0$, which gives $\nabla^2(g) = 0$. This means that $\Theta_B \in \Lambda^2 M \otimes \mathfrak{u}_B$, **and this for is real with respect to the real structure defined by $\iota$.**

**Step 3:** The $(0,2)$-part of the curvature vanishes, because $\overline{\partial}^2 = 0$. The $(2,0)$-part of the curvature vanishes, because $\iota(\Theta_B) = \Theta_B$, and **any real structure on $\operatorname{End}(B)$ exchanges $\Lambda^{2,0}(M) \otimes \operatorname{End}(B)$ and $\Lambda^{0,2}(M) \otimes \operatorname{End}(B)$.**

■

**COROLLARY: For the Chern connection $\nabla = \overline{\partial} + \nabla^{1,0}$ on $B$, one has $\Theta_B = \{\nabla^{1,0}, \overline{\partial}\}$.**

**COROLLARY: The curvature of a holomorphic Hermitian line bundle is a closed (1,1)-form.**

## Curvature of the Chern connection on a line bundle

**REMARK:** Let $B$ he a Hermitian holomorphic line bundle, and $b \in \Gamma(B)$ a nowhere vanishing holomorphic section. Then

$$d|b|^2 = (\nabla^{1,0}b, b) + (b, \nabla^{1,0}b) = 2\,\mathrm{Re}(\nabla^{1,0}b, b),$$

which gives $\nabla^{1,0}b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b$. **We obtain that $\Theta_B(b) = 2\bar\partial\partial \log |b| b$, hence $\Theta_B = -2\partial\bar\partial \log |b|$.**

**REMARK:** The same formula holds for $B$ of any rank. Consider a holomorphic section $b$ of $B$, and let $(\Theta(b), b)$ be the corresponding 2-form on $M$. **Then $(\Theta(b), b)(x, y) = \bar\partial\partial |b|^2(x, y)$.**

**REMARK:** Suppose that $b = fb_1$, where $f$ is a holomorphic function. Then

$$\partial\bar\partial \log |b| = \partial\bar\partial \log |b_1| + \partial\bar\partial \log |f| = \partial\bar\partial \log |b_1| + \frac{1}{2}\partial\bar\partial \log(f\bar f) =$$
$$= \partial\bar\partial \log |b_1| + \frac{1}{2}\partial\bar\partial \log(f) + \frac{1}{2}\partial\bar\partial \log(\bar f).$$

The last two terms vanish because $\bar\partial \log f = 0$ and $\partial \log \bar f = 0$, hence $\partial\bar\partial \log |b| = \partial\bar\partial \log |b_1|$. This implies that the curvature 2-form $\partial\bar\partial \log |b|$ **is independent from the choice of a holomorphic section $b$.**