Complex surfaces

lecture 4: Locally conformall Kähler manifolds

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Algebraic cones and Vaisman manifolds (reminder)

DEFINITION: Let P be a projective orbifold, and L an ample line bundle on P. Assume that the total space $\operatorname{Tot}^\circ(L)$ of all non-zero vectors in L is smooth. **An open algebraic cone** is $\operatorname{Tot}^\circ(L)$.

EXAMPLE: Let $P \subset \mathbb{C}P^n$, and $L = \mathcal{O}(1)|_P$. Then **the open algebraic cone** $\operatorname{Tot}^\circ(L)$ **can be identified with the set** $\pi^{-1}(P)$ of all $v \in \mathbb{C}^{n+1}\backslash 0$ projected to P under the standard map $\pi: \mathbb{C}^{n+1}\backslash 0 \to \mathbb{C}P^n$. **The closed algebraic cone** is its closure in $\mathbb{C}^{n+1}$. It is an affine subvariety in $\mathbb{C}^{n+1}$ given by the same collection of homogeneous equations as P. Its **origin** is zero.

DEFINITION: An automorphism $A: P \longrightarrow P$ is **L-linearizable** L admits an A-equivariant structure, in other words, if A can be lifted to an automorphism of the cone $\operatorname{Tot}^\circ(L)$ which is linear on fibers.

DEFINITION: Fix a Hermitian metric on L, and let $A: \operatorname{Tot}^\circ(L) \longrightarrow \operatorname{Tot}^\circ(L)$ be an automorphism which is linear on fibers and satisfies $|A(v)| = \lambda |v|$ for some number $\lambda < 1$. Then A acts on the closed cone as a holomorphic contraction, and the quotient space $\operatorname{Tot}^\circ(L)/\langle A\rangle$ is Hausdorff for the same reason as it is Hausdorff for a Hopf manifold (an action of an invertible contraction is always totally discontinuous outside of the origin; **prove this as an exercise).** The quotient manifold $\operatorname{Tot}^\circ(L)/\langle A\rangle$ is called **a Vaisman manifold**.

LCK manifolds in terms of differential forms

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M \geqslant 2$. Then M is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form**.

REMARK: This condition is **conformally invariant**, that is, preserved if we replace ω by a conformally equivalent form $f\omega$, where $f > 0$ is a positive smooth function. Indeed,

$$d(f\omega) = df \wedge \omega + f d\omega = df \wedge \omega + f\theta \wedge \omega = (df + f\theta) \wedge \omega.$$

EXAMPLE: Let $\tilde{\omega} = \sum_i dx_i \wedge dy_i$ be the standard flat Hermitian form on $\mathbb{C}^n$, and $\psi(z) := |z|^2$. Clearly, **the form $\frac{\tilde{\omega}}{\psi}$ is homothety invariant.** Consider the classical Hopf manifold $\frac{\mathbb{C}^n \backslash 0}{\langle \lambda \cdot \operatorname{Id} \rangle}$, and a Hermitian metric $\omega := \frac{\tilde{\omega}}{\psi}$ on H. **Since ω is conformally equivallent to Kähler form, it is LCK.**

EXAMPLE: Any Vaisman manifold (defined, as in Lecture 2, as a quotient of an algebraic cone) **is LCK**, as explained later in this lecture.

Chern connection

DEFINITION: Let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}\ B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1}: V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0}: V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: An Hermitian holomorphic vector bundle is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure operator $\overline{\partial}$.

DEFINITION: A Chern connection on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

Curvature of a holomorphic line bundle

REMARK: If L is a line bundle, $\operatorname{End} L$ is trivial, and **the curvature Θ_L of L is a closed 2-form.**

REMARK: When speaking of a **"curvature of a holomorphic bundle",** one usually means the curvature of a Chern connection.

REMARK: Let B be a holomorphic Hermitian line bundle, and b its non-degenerate holomorphic section. Denote by η a (1,0)-form which satisfies $\nabla^{1,0} b = \eta \otimes b$. Then $d|b|^2 = \operatorname{Re} g(\nabla^{1,0} b, b) = \operatorname{Re} \eta |b|^2$. **This gives** $\nabla^{1,0} b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b.$

REMARK: Then $\Theta_B(b) = 2\bar\partial\partial \log |b| b$, **that is,** $\Theta_B = -2\partial\bar\partial \log |b|$**.**

COROLLARY: If $g' = e^{2f} g$ – two metrics on a holomorphic line bundle, Θ, Θ' their curvatures, **one has** $\Theta' - \Theta = -2\partial\bar\partial f$

Calabi's formula (2.6)

REMARK: This formula has no better name that "Calabi's (2.6)", from *E. Calabi, Métriques kählériennes et fibrés holomorphes, Ann. Ecole Norm. Sup.* **12** *(1979), 269-294.* We formulate it for line bundles; in this situation it is easier to state.

PROPOSITION: Let L be a holomorphic Hermitian line bundle on a complex manifold M, and $\Theta \in \Lambda^{1,1}(M)$ the curvature of its Chern connection. Consider the function $\psi \in C^\infty \operatorname{Tot}(L)$, $v \stackrel{\psi}{\mapsto} |v|^2$, and let $\pi: \operatorname{Tot}(L) \longrightarrow M$ be the projection map. Using the Chern connection on $\operatorname{Tot}(L)$, we decompose $T\operatorname{Tot}(L) = \ker d\pi \oplus T_{\mathrm{hor}} \operatorname{Tot}(L)$; here $\ker d\pi$ is fiberwise (vertical) tangent vectors, and $T_{\mathrm{hor}} \operatorname{Tot}(L)$ the bundle of horizontal tangent vectors, $T_{\mathrm{hor}} \operatorname{Tot}(L) \cong \pi^* TM$. Denote by ω_π the Hermitian form on fibers of L, which is set to zero on $T_{\mathrm{hor}} \operatorname{Tot}(L)$. **Then**

$$dd^c\psi = -\psi\pi^*\Theta + \omega_\pi.$$

In particular, **when L^* is ample and the curvature of L is positive, the function ψ is plurisubharmonic, and strictly plurisubharmonic on $\operatorname{Tot}^\circ(L)$.**

Proof: https://arxiv.org/abs/2208.07188 *(Liviu Ornea, Misha Verbitsky, Principles of Locally Conformally Kahler Geometry, Theorem 5.30).* ■

Vaisman manifolds are LCK

COROLLARY: Let L be a line bundle with negative curvature on a projective manifold. Then the form $\frac{dd^c\psi}{\psi}$ **is homothety invariant and locally conformally Kähler on** $\operatorname{Tot}^\circ(L)$**.** Moreover, **this form is a pullback of an LCK form on** $\frac{\operatorname{Tot}^\circ(L)}{\langle A\rangle}$**,** where A acts on $\operatorname{Tot}(L)$ as in the definition of a Vaisman manifold.

Proof: Homothety with coefficient λ preserves $\frac{dd^c\psi}{\psi}$, because it multiplies numerator and denominator by $|\lambda|^2$. This form is Hermitian because $dd^c\psi = -\psi\pi^*\Theta + \omega_\pi$; the first component is positive on horizontal vectors and vanishes on vertical vectors, the second positive on vertical vectors and vanishes on horizontal. ■

REMARK: We have just shown that **Vaisman manifolds are LCK.**

Homotheties and monodromy

REMARK: In a few slides, we are going to give a definition of LCK manifold in terms of a Kähler form on the universal covering, or Kähler forms taking values in a local system, and prove the equivalence of these definitions. Under the alternative definition, "LCK manifold" is a quotient of a Kähler manifold by a free action of cocompact, discrete group acting by homotheties.

DEFINITION: **Deck transform maps**, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$ (prove this as an exercise).**

CLAIM: Any conformal map $\varphi : (M, \omega) \longrightarrow (M_1, \omega_1)$ of Kähler manifolds **is a homothety.**

Proof: By definition, there exists a function $f > 0$ such that $\varphi^* \omega_1 = f\omega$; **we need to show that $f = const$.** However, $0 = d(\varphi^* \omega_1) = df \wedge \omega$. Since $\Lambda^1 M \xrightarrow{\wedge\omega} \Lambda^3 M$ is injective **(check this),** this implies that $df = 0$. ∎

Curvature of a connection (reminder)

DEFINITION: Let $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a smooth budnle. Extend it to an operator on B-valued forms

$$B \xrightarrow{\nabla} \Lambda^1(M) \otimes B \xrightarrow{\nabla} \Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B \xrightarrow{\nabla} \dots$$

using $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde{\eta}}\eta \wedge \nabla b$. The operator $\nabla^2: B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇. The operator $\nabla: \Lambda^i(M) \otimes B \xrightarrow{\nabla} \Lambda^{i+1}(M) \otimes B$ **is denoted d_∇.**

REMARK: $d_{\nabla_0} = d$ if B is a trivial bundle with the trivial connection ∇_0. When $\nabla = \nabla_0 + \wedge\theta$, where $\wedge\theta$ is the multiplication by a 1-form θ, we have $d_\nabla(\eta) = d\eta + \theta \wedge \eta$.

REMARK: The algebra of $\operatorname{End}(B)$-valued forms naturally acts on $\Lambda^* M \otimes B$. The curvature satisfies $\nabla^2(fb) = d^2 fb + df \wedge \nabla b - df \wedge \nabla b + f\nabla^2 b = f\nabla^2 b$, hence it is $C^\infty M$-linear. **We consider it as an $\operatorname{End}(B)$-valued 2-form on M.** A connection is **flat** if its curvature vanishes.

Local systems and Riemann-Hilbert correspondence

DEFINITION: A **local system** on a manifold is a locally constant sheaf of vector spaces.

THEOREM: Fix a point $x \in M$. Then **the category of local systems is naturally equivalent to the category of representations of $\pi_1(M, x)$.**
Proof: `http://verbit.ru/IMPA/RS-2024/slides-RS-2024-17.pdf`, pages 5-8. ■

THEOREM: The category of vector bundles with flat connection **is equivalent to the category of local systems.**
Proof. Step 1: See `http://verbit.ru/IMPA/RS-2024/slides-RS-2024-20.pdf`. From a locally constant sheaf $\mathbb{V}$ we obtain a vector bundle $B := \mathbb{V} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$, where $\mathbb{R}_M$ is the constant sheaf on M. If $v_1, ..., v_n$ is a basis in $\mathbb{V}(U)$, all sections of $B(U)$ have a form $\sum_{i=1}^n f_i v_i$, where $f_i \in C^\infty U$. Define the connection ∇ by $\nabla\left(\sum_{i=1}^n f_i v_i\right) = \sum df_i \otimes v_i$. This connection is flat because $d^2 = 0$. It is independent from the choice of v_i because v_i is defined canonically up to a matrix with constant coefficients. **We have constructed a functor from locally constant sheaves to flat vector bundles**.

Step 2: The converse functor takes a flat bundle (B, ∇) on M to the sheaf of parallel sections of B; this sheaf is locally constant, because every vector can be locally extended to a parallel section uniquely (the proof of this non-trivial observation relies on Frobenius theorem). ■

χ-automorphic forms

DEFINITION: Let $\tilde{M} \xrightarrow{\pi} M$ be the universal covering of M, and $\chi: \pi_1(M) \longrightarrow \mathbb{R}^{>0}$ a character (group homomorphism). Consider the natural action of $\pi_1(M)$ on $\tilde{M}$ **An χ-automorphic form** on $\tilde{M}$ is a differential form $\eta \in \Lambda^k(\tilde{M})$ which satisfies $\gamma^*\eta = \chi(\gamma)\eta$ for any $\gamma \in \pi_1(M)$.

Proposition 1: Let L be a rank 1 local system on M associated with the representation χ. Then **the space of χ-automorphic k-forms on $\tilde{M}$ is in natural correspondence with the space of sections of $\Lambda^k(M) \otimes L$.** Under this equivalence, **the de Rham differential on χ-automorphic forms corresponds to the operator $d_\nabla: \Lambda^k(M) \otimes L \longrightarrow \Lambda^{k+1}(M) \otimes L$.**

Proof. Step 1: Consider the pullback $\tilde{L} := \pi^* L$. The bundle $\tilde{L}$ is flat and has trivial monodromy, hence it is naturally trivialized by parallel sections. This identifies $\tilde{L}$ with $C^\infty M$; however, such an identification is defined up to a real constant. Choosing the constant, we fix the map $\psi: L \longrightarrow \tilde{C}^\infty M$. The bundle $\tilde{L}$ is $\pi_1(M)$-equivatiant, with $\gamma \in \pi_1(M)$ taking a parallel section $v \in H^0(\tilde{L})$ to $\chi(\gamma)v$. This implies that for any section u of L, the pullback $\tilde{u} \in H^0(\tilde{L})$ satisfies $\gamma^*(\psi(\tilde{u})) = \chi(\gamma)\psi(\tilde{u})$. **This identifies sections of L with χ-automorphic functions on $\tilde{M}$.**

Step 2: Since $u \mapsto \psi(\pi^*(u))$ takes a parallel section of L to a constant function, this correspondence takes d_∇ to de Rham differential. ∎

Lichnerowicz cohomology

Let (L,∇) be a real flat line bundle. Any such bundle is trivialized; let ∇_0 be the trivial connection, and $\nabla-\nabla_0\in\Lambda^1 M$ the corresponding 1-form. Since $(\nabla_0+\theta)^2 = d_{\nabla_0}(\theta)=0$, the 1-form is closed, and the differential d_∇ is equal to $d+\wedge\theta$.

DEFINITION: Let θ be a closed 1-form on a manifold, and $d_\theta(\alpha):=d\alpha+\theta\wedge\alpha$ be the corresponding differential on $\Lambda^*(M)$. Its cohomology are called **Morse-Novikov cohomology**, or **Lichnerowicz cohomology**.

THEOREM: Lichnerowitz cohomology of a manifold **is equal to the cohomology with coefficients in a local system defined by (L,∇).**

Proof: $L \xrightarrow{d_\theta} L\otimes\Lambda^1 M \xrightarrow{d_\theta} L\otimes\Lambda^2 M \xrightarrow{d_\theta}$ ... is fine resolution of the sheaf of parallel sections of L. ■

LCK manifolds in terms of an L-valued Kähler form

DEFINITION: Let (L, ∇) be an oriented real line bundle with flat connection on a complex manifold M, and $\omega \in L \otimes \Lambda^{1,1}M$ a (1,1)-form with values in L. We say that ω is an **L-valued Kähler form** if $\omega(x, Ix) \in L$ is (strictly) positive for any non-zero tangent vector, and $d_\nabla \omega = 0$.

REMARK: If we use a trivialization to identify L and $C^\infty M$, ω becomes a (1,1)-form, and d_∇ becomes d_θ, giving $d_\nabla(\alpha) = d\alpha + \theta \wedge \alpha$. Therefore, **$L$-valued Kähler form on a manifold is the same as an LCK-form.**

LCK manifolds in terms of deck transform

Another definition: An LCK manifold is a complex manifold M, $\dim_{\mathbb{C}} M \geqslant 2$ such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

THEOREM: These two definitions are equivalent.

Proof. Step 1: Let $\tilde{\omega}$ be an automorphic Kähler form on $\tilde{M}$, $\chi: \pi_1(M) \longrightarrow \mathbb{R}^{>0}$ be the character taking γ to the number $\frac{\gamma^*\tilde{\omega}}{\tilde{\omega}}$, and (L, ∇) the corresponding flat line bundle on M. By Proposition 1, the automorphic Kähler form $\tilde{\omega}$ on M corresponds to a d_∇-closed form $\omega \in \Lambda^{1,1}(M) \otimes L$. Any trivialization of L produces a trivial connection ∇_0 such that $\nabla - \nabla_0(f) = f\theta$ for some 1-form θ. Then $d_{\nabla_0}(\omega) = d_{\nabla_0} - d_\nabla(\omega) = \omega \wedge \theta$. However, d_{∇_0} is de Rham differential, which brings $d\omega = \omega \wedge \theta$.

Step 2: Conversely, assume $d\omega = \omega \wedge \theta$, where θ is a closed 1-form. The connection $\nabla_0 - \wedge\theta$ on the trivial line bundle L is flat, because $d\theta = 0$. Then $d_\nabla \omega = 0$, which allows one to lift ω to an automorphic Kähler form on $\tilde{M}$ using Proposition 1. ■

Next lecture:

Vaisman theorem

definitions of Vaisman manifolds (Vaisman definition, Kamishima-Ornea, Istrati)

Vaisman manifolds as quotients of algebraic cones (proper, improper)

Canonical foliation, subvarieties of Vaisman manifolds