Complex surfaces

lecture 5: Local systems and locally conformally Kähler manifolds

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Homotheties and monodromy

REMARK: Today I am going to give two equivalent definitions of LCK manifolds, one in terms of a Kähler form on the universal cover, and another in terms of Kähler forms taking values in a local system. Under the first of these definitions, "LCK manifold" is a quotient of a Kähler manifold by a free action of cocompact, discrete group acting by homotheties.

DEFINITION: **Deck transform maps**, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$ (prove this as an exercise).**

CLAIM: Any conformal map $\varphi : (M, \omega) \longrightarrow (M_1, \omega_1)$ of Kähler manifolds **is a homothety.**

Proof: By definition, there exists a function $f > 0$ such that $\varphi^* \omega_1 = f\omega$; **we need to show that $f = const$.** However, $0 = d(\varphi^*\omega_1) = df \wedge \omega$. Since $\Lambda^1 M \xrightarrow{\wedge\omega} \Lambda^3 M$ is injective **(check this),** this implies that $df = 0$. ■

Curvature of a connection (reminder)

DEFINITION: Let $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a smooth budnle. Extend it to an operator on B-valued forms

$$B \xrightarrow{\nabla} \Lambda^1(M) \otimes B \xrightarrow{\nabla} \Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B \xrightarrow{\nabla} ...$$

using $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde\eta}\eta \wedge \nabla b$. The operator $\nabla^2: B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇. The operator $\nabla: \Lambda^i(M) \otimes B \xrightarrow{\nabla} \Lambda^{i+1}(M) \otimes B$ **is denoted d_∇.**

REMARK: $d_{\nabla_0} = d$ if B is a trivial bundle with the trivial connection ∇_0. When $\nabla = \nabla_0 + \wedge\theta$, where $\wedge\theta$ is the multiplication by a 1-form θ, we have $d_\nabla(\eta) = d\eta + \theta \wedge \eta$.

REMARK: The algebra of End(B)-valued forms naturally acts on $\Lambda^* M \otimes B$. The curvature satisfies $\nabla^2(fb) = d^2 fb + df \wedge \nabla b - df \wedge \nabla b + f\nabla^2 b = f\nabla^2 b$, hence it is $C^\infty M$-linear. **We consider it as an End(B)-valued 2-form on M.** A connection is **flat** if its curvature vanishes.

Local systems and Riemann-Hilbert correspondence

DEFINITION: A **local system** on a manifold is a locally constant sheaf of vector spaces.

THEOREM: Fix a point $x \in M$. Then **the category of local systems is naturally equivalent to the category of representations of $\pi_1(M,x)$.**
Proof: `http://verbit.ru/IMPA/RS-2024/slides-RS-2024-17.pdf`, pages 5-8. ■

THEOREM: The category of vector bundles with flat connection **is equivalent to the category of local systems.**
Proof. Step 1: See `http://verbit.ru/IMPA/RS-2024/slides-RS-2024-20.pdf`. From a locally constant sheaf $\mathbb{V}$ we obtain a vector bundle $B := \mathbb{V} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$, where $\mathbb{R}_M$ is the constant sheaf on M. If $v_1, ..., v_n$ is a basis in $\mathbb{V}(U)$, all sections of $B(U)$ have a form $\sum_{i=1}^n f_i v_i$, where $f_i \in C^\infty U$. Define the connection ∇ by $\nabla\left(\sum_{i=1}^n f_i v_i\right) = \sum df_i \otimes v_i$. This connection is flat because $d^2 = 0$. It is independent from the choice of v_i because v_i is defined canonically up to a matrix with constant coefficients. **We have constructed a functor from locally constant sheaves to flat vector bundles**.

Step 2: The converse functor takes a flat bundle (B, ∇) on M to the sheaf of parallel sections of B; this sheaf is locally constant, because every vector can be locally extended to a parallel section uniquely (the proof of this non-trivial observation relies on Frobenius theorem). ■

χ-automorphic forms

DEFINITION: Let $\tilde{M} \xrightarrow{\pi} M$ be the universal covering of M, and $\chi : \pi_1(M) \longrightarrow \mathbb{R}^{>0}$ a character (group homomorphism). Consider the natural action of $\pi_1(M)$ on $\tilde{M}$ **An χ-automorphic form** on $\tilde{M}$ is a differential form $\eta \in \Lambda^k(\tilde{M})$ which satisfies $\gamma^*\eta = \chi(\gamma)\eta$ for any $\gamma \in \pi_1(M)$.

Proposition 1: Let L be a rank 1 local system on M associated with the representation χ. Fix a smooth trivialization of L. Then **the space of χ-automorphic k-forms on $\tilde{M}$ is in a natural correspondence with the space of sections of $\Lambda^k(M) \otimes L$.** Under this equivalence, **the de Rham differential on χ-automorphic forms corresponds to the operator $d_\nabla : \Lambda^k(M) \otimes L \longrightarrow \Lambda^{k+1}(M) \otimes L$.**

Proof. Step 1: Let u_1 be a nowhere vanishing section of L, and θ a 1-form such that $\nabla u_1 = u_1 \otimes \theta$. Since ∇ is flat, θ is closed. Given a path $A : S^1 \longrightarrow M$, the monodromy of (L, ∇) along A is equal to $\exp(\int_A \theta)$. **Therefore, $\pi^*\theta = d\log(\psi)$, where ψ is a everywhere positive χ-automorphic function on $\tilde{M}$.**

χ-automorphic forms (2)

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Step 2: Given a section fu_1 of L, $f\in C^\infty M$, we define $\sigma(fu_1) := \tilde{f}\psi$, where $\tilde{f} = \pi^* f$. **This correspondence takes ∇ to de Rham differential,** in the following sense: $d(\sigma(fu_1)) = \sigma_1(\nabla(fu_1))$, where $\sigma_1(\alpha\otimes l) := \pi^*\alpha\sigma(l)$. Indeed, for any section fu_1 of L we have

$$d(\sigma(fu_1)) = d(\tilde{f}\psi) = \psi d\tilde{f} + \tilde{f}\psi\frac{d\psi}{\psi} = \sigma_1(df\otimes u_1) + \sigma_1(f\nabla(u_1)) = \sigma_1(\nabla(fu_1)).$$

Step 3: Let now $s: \Lambda^* M\otimes L: \longrightarrow \Lambda^*\tilde{M}$ take $\eta\otimes u_1$ to $\psi\pi^*\eta$. By Step 1, this map satisfies $s(d_\nabla(\eta\otimes u_1)) = d(s(\eta\otimes u_1))$. **This gives a bijective correspondence between sections of $\Lambda^* M\otimes L$ and χ-automorphic forms on $\tilde{M}$.** ∎

Lichnerowicz cohomology

Let (L, ∇) be a real flat line bundle. Any such bundle is trivialized; let ∇_0 be the trivial connection, and $\nabla - \nabla_0 \in \Lambda^1 M$ the corresponding 1-form. Since $(\nabla_0 + \theta)^2 = d_{\nabla_0}(\theta) = 0$, the 1-form is closed, and the differential d_∇ is equal to $d + \wedge\theta$.

DEFINITION: Let θ be a closed 1-form on a manifold, and $d_\theta(\alpha) := d\alpha + \theta\wedge\alpha$ be the corresponding differential on $\Lambda^*(M)$. Its cohomology are called **Morse-Novikov cohomology**, or **Lichnerowicz cohomology**, denoted $H^*_\theta(M)$.

THEOREM: Lichnerowitz cohomology of a manifold **is equal to the cohomology with coefficients in a local system defined by (L, ∇).**

Proof: $L \xrightarrow{d_\theta} L \otimes \Lambda^1 M \xrightarrow{d_\theta} L \otimes \Lambda^2 M \xrightarrow{d_\theta}$ ... is a fine resolution of the sheaf of parallel sections of L. ∎

LCK manifolds in terms of an L-valued Kähler form

DEFINITION: Let (L, ∇) be an oriented real line bundle with flat connection on a complex manifold M, and $\omega \in L \otimes \Lambda^{1,1}M$ a (1,1)-form with values in L. We say that ω is an **L-valued Kähler form** if $\omega(x, Ix) \in L$ is (strictly) positive for any non-zero tangent vector, and $d_\nabla \omega = 0$.

REMARK: If we use a trivialization to identify L and $C^\infty M$, ω becomes a (1,1)-form, and d_∇ becomes d_θ, giving $d_\nabla(\alpha) = d\alpha + \theta \wedge \alpha$. Therefore, **$L$-valued Kähler form on a manifold is the same as an LCK-form.**

LCK manifolds in terms of deck transform

Another definition: An LCK manifold is a complex manifold M, $\dim_{\mathbb{C}} M \geqslant 2$ such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

THEOREM: These two definitions are equivalent.

Proof. Step 1: Let $\tilde{\omega}$ be an automorphic Kähler form on $\tilde{M}$, $\chi: \pi_1(M) \longrightarrow \mathbb{R}^{>0}$ be the character taking γ to the number $\frac{\gamma^*\tilde{\omega}}{\tilde{\omega}}$, and (L, ∇) the corresponding flat line bundle on M. By Proposition 1, the automorphic Kähler form $\tilde{\omega}$ on M corresponds to a d_∇-closed form $\omega \in \Lambda^{1,1}(M) \otimes L$. Any trivialization of L produces a trivial connection ∇_0 such that $\nabla - \nabla_0(f) = f\theta$ for some 1-form θ. Then $d_{\nabla_0}(\omega) = d_{\nabla_0} - d_\nabla(\omega) = \omega \wedge \theta$. However, d_{∇_0} is de Rham differential, which brings $d\omega = \omega \wedge \theta$.

Step 2: Conversely, assume $d\omega = \omega \wedge \theta$, where θ is a closed 1-form. The connection $\nabla_0 - \wedge\theta$ on the trivial line bundle L is flat, because $d\theta = 0$. Then $d_\nabla \omega = 0$, which allows one to lift ω to an automorphic Kähler form on $\tilde{M}$ using Proposition 1. ■

Next lecture:

Vaisman theorem

definitions of Vaisman manifolds (Vaisman definition, Kamishima-Ornea, Istrati)

Vaisman manifolds as quotients of algebraic cones (proper, improper)

Canonical foliation, subvarieties of Vaisman manifolds