Complex surfaces

lecture 6: Vaisman theorem

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LCK manifolds (reminder)

DEFINITION: A complex Hermitian manifold of dimension $\dim_{\mathbb{C}} > 1$ (M, I, g, ω) is called **locally conformally Kähler** (LCK) if there exists a closed 1-form θ such that $d\omega = \theta \wedge \omega$. The 1-form θ is called the **Lee form**m and its cohomology class **the Lee class**.

REMARK: This definition **is equivalent to the existence of a Kähler cover $(\tilde{M}, \tilde{\omega}) \longrightarrow M$ such that the deck group Γ acts on $(M, \tilde{\omega})$ by holomorphic homotheties.** Indeed, suppose that θ is exact, $df = \theta$. **Then $e^{-f}\omega$ is a Kähler form.** Let $\tilde{M}$ be a covering such that the pullback $\tilde{\theta}$ of θ is exact, $df = \tilde{\theta}$. Then the pullback of $\tilde{\omega}$ is conformal to the Kähler form $e^{-f}\tilde{\omega}$.

Vaisman theorem

REMARK: Let (M, ω, θ) be an LCK manifold, and θ' another 1-form, homologous to θ. Write $\theta' - \theta = df$. Then

$$d(e^f \omega) = e^f(d\omega + df \wedge \omega) = e^f(\theta \wedge \omega + df \wedge \omega) = \theta' \wedge (e^f \omega).$$

In other words, **conformally equivalent LCK metric give rise to homologous Lee forms, and any closed 1-form cohomologous to the Lee form is a Lee form of a conformally equivalent LCK metric.**

DEFINITION: A compact manifold admitting a Kähler form is called **of Kähler type**.

THEOREM: (Vaisman)
A compact LCK manifold (M, I, θ) with non-exact Lee form **does not admit a Kähler structure.**

Proof: On a compact manifold of Kähler type, any $[\theta] \in H^1(M, \mathbb{R})$ can be represented by α, obtained as a real part of a holomorphic form. This gives $d^c\alpha = 0$. After a conformal change of the metric, we can assume that $d\omega = \alpha \wedge \omega$, and $dd^c\omega = \alpha \wedge I(\alpha) \wedge \omega$. **On a Kähler manifold, a positive exact form must vanish, which implies $\alpha \wedge I(\alpha) \wedge \omega = 0$ and $\alpha = 0$.** ■

REMARK: Such manifolds are called **strict LCK**. Speaking of compact LCK manifolds, people usually make this assumption tacitly.

Izu Vaisman

Izu Vaisman, b. June 22, 1938 in Jassy, Romania

Vaisman manifolds, more definitions

DEFINITION: Let (M, ω, θ) be an LCK manifold. It is called **Vaisman** if $\nabla\theta = 0$, where ∇ is the Levi-Civita connection of the Hermitian metric.

REMARK: In Lecture 2, I gave another definition of Vaisman manifold, which is equivalent to the one above. A third definition, also equivalent:

DEFINITION: An LCK manifold is a **Vaisman manifold** if it admits a continuous, holomorphic, conformal action of a complex Lie group.

REMARK: Equivalence of these three definitions is a **highly non-trivial theorem;** see "Principles of LCK geometry" for all proofs.

Vaisman manifolds, examples and non-examples

EXAMPLE: All non-Kähler elliptic surfaces are Vaisman.

DEFINITION: A linear Hopf manifold is a quotient $M := \frac{\mathbb{C}^n \backslash 0}{\langle A \rangle}$ where A is a linear contraction. When A is diagonalizable, M is called **diagonal Hopf.**

EXAMPLE: All diagonal Hopf manifolds are Vaisman, and **when A cannot be diagonalized, M is LCK and not Vaisman**.

THEOREM: (Ornea-V.)
All complex submanifolds of Vaisman manifolds are Vaisman. All Vaisman manifolds admit a holomorphic embedding to a diagonal Hopf manifold (which is Vaisman, too).

The fundamental foliation

DEFINITION: Let M be a Vaisman manifold, $\theta^\sharp$ its Lee field, and Σ a 2-dimensional real foliation generated by $\theta^\sharp, I\theta^\sharp$. It is called **the fundamental foliation** of M. By construction, Σ is tangent to the orbits of the one-parametric complex Lie group of automorphisms of the covering $\tilde{M}$. Therefore, **Σ is a holomorphic foliaton.**

REMARK: Let $M = \operatorname{Tot}^\circ(L)/\mathbb{Z}$ be a quotient of an algebraic cone. **Then Σ is tangent to the fibers of the bundle L.**

THEOREM: Let M be a compact Vaisman manifold, and $\Sigma \subset TM$ its fundamental foliation. Then

1. **Σ is independent from the choice of the Vaisman metric.**
2. **There exists a positive, exact (1,1)-form ω_0 with $\Sigma = \ker \omega_0$.**
3. **For any complex subvariety $Z \subset M$, Z is tangent to Σ.**
4. **For any compact complex subvariety $Z \subset M$, the set of smooth points of Z is Vaisman.**