Complex surfaces

lecture 7: Elliptic operators of second order

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Differential operators

DEFINITION: Let M be a smooth manifold **The ring of differential operators** Diff(M) is the ring of maps $C^\infty M \longrightarrow C^\infty M$ generated by vector fields and maps $L_f(\alpha) := f\alpha$ for all $f \in C^\infty M$.

DEFINITION: Clearly, $L_f L_g = L_{fg}$, hence such map generate a ring isomorphic to $C^\infty M$. It is called **the ring of differential operators of order 0**. **Differential operators of order d** are operators obtained by composition of at most

REMARK: Vector fields on M **are the same as derivations of the ring $C^\infty M$**, hence Diff(M) is an associative ring generated by $C^\infty M$ and derivations.

REMARK: Locally, an order d differential operator can be written as a sum of **differential monomials**:

$$D = f_0 + \sum_{i=1}^{d} f_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{n} f_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j,k=1}^{n} f_{ijk} \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} + ...$$

Associated graded rings

REMARK: **Algebra** is an associative ring over a field. Rings in this lecture **are not necessarily commutative, but always associative.**

DEFINITION: Let R be an associative ring. **Filtration** on R is a collection of subspaces $R_0 \subset R_1 \subset R_2 \subset ...$ such that $R_i R_j \subset R_{i+j}$.

REMARK: Let $x \in R_k, y \in R_l$. Then the product xy modulo R_{k+l-1} depends only on the class of x modulo R_{k-1}. Indeed, $R_{k-1}R_l \subset R_{k+l-1}$. **This defines the product map $(R_k/R_{k-1}) \otimes (R_l/R_{l-1}) \longrightarrow R_{k+l}/R_{k+l-1}$. We obtained the associative product structure on the space $\bigoplus_{i=0}^{\infty} R_i/R_{i-1}$.**

DEFINITION: Let $R_0 \subset R_1 \subset R_2 \subset ...$ be a filtered ring. The ring $\bigoplus_{i=0}^{\infty} R_i/R_{i-1}$ is called **the associated graded ring** of this filtration.

EXAMPLE: Consider the filtration $\operatorname{Diff}^0(M) \subset \operatorname{Diff}^1(M) \subset \operatorname{Diff}^2(M) \subset ...$ on the ring of differential operators. The associated graded ring is called **the ring of symbols of differential operators**.

The ring of symbols

THEOREM: Consider the filtration $\operatorname{Diff}^0(M) \subset \operatorname{Diff}^1(M) \subset \operatorname{Diff}^2(M) \subset ...$ on the ring of differential operators. Then **its associated graded ring is isomorphic to the ring $\bigoplus_i \operatorname{Sym}^i(TM)$.**

Proof. Step 1: Let f be a function with zero of order $\geqslant k$ in z, and $\mathfrak{m}$ the maximal ideal of z. Then $\operatorname{Diff}^{k-1}(f)$ vanishes at z. This gives a bilinear pairing $\frac{\operatorname{Diff}^k(M)}{\operatorname{Diff}^{k-1}(M)} \otimes \frac{\mathfrak{m}^k}{\mathfrak{m}^{k-1}} \longrightarrow \mathbb{R}$, mapping $D \otimes f$ to $Df(0)$. Since $(\mathfrak{m}^k)/(\mathfrak{m}^{k-1}) = \operatorname{Sym}^i(T_zM)^*$, this pairing, applied to all $z \in M$, gives a natural map $\operatorname{Diff}^k(M)/\operatorname{Diff}^{k-1}(M) \xrightarrow{\sigma} \operatorname{Sym}^k(TM)$. **It remains to prove that σ is an isomorphism.** Since this pairing is local, it suffices to prove that it is an isomorphism for $M = \mathbb{R}^n$.

Step 2: Let $x_1, ..., x_n$ be coordinates in $M = \mathbb{R}^n$, and $D \in \operatorname{Diff}^k(M)$. Then $\operatorname{Sym}^*(TM) = C^\infty M[\frac{d}{dx_1}, \frac{d}{dx_2}, ..., \frac{d}{dx_n}]$. Consider a homogeneous differential monomial $D = f\frac{\partial^k}{\partial x_{i_1} ... \partial x_{i_k}}$. Then $\sigma(D) = f\frac{d}{dx_{i_1}}\frac{d}{dx_{i_2}} ... \frac{d}{dx_{i_k}}$. **Therefore, σ is surjective.**

Step 3: Let $D \in \operatorname{Diff}^k(\mathbb{R}^n)$ be a differential operator $\sigma(D) = 0$. Expressing D as a sum of differential monomials $D = f_0 + \sum_{i=1}^d f_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^n f_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j,k=1}^n f_{ijk} \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} + ...$, we obtain that $\sigma(D)$ is the sum of those monomials which have of degree k. **Therefore, $\sigma(D) = 0$ implies $D \in \operatorname{Diff}^{k-1}(\mathbb{R}^n)$.** ∎

Elliptic operators

COROLLARY: Consider the filtration $\mathrm{Diff}^0(M) \subset \mathrm{Diff}^1(M) \subset \mathrm{Diff}^2(M) \subset$ .... Then **its associated graded ring is isomorphic to $\bigoplus_i \mathrm{Sym}^i(TM)$,** identified with the ring if fiberwise polynomial functions on T^*M. ■

DEFINITION: Let D be a differential operator of order k. Its **symbol** is the image of D in $\frac{\mathrm{Diff}^k(M)}{\mathrm{Diff}^{k-1}(M)} = \mathrm{Sym}^k(TM)$.

DEFINITION: Let D be a differential operator of order k, and $\mathrm{symb}(D) \in \mathrm{Sym}^k(TM)$ its symbol. Consider $\mathrm{symb}(D)$ as a degree k polynomial function on T^*M. We say that **D is elliptic** if $\mathrm{symb}(D)$ is positive or negative everywhere on $T^*M\backslash 0$.

REMARK: Let $D = f_0 + \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^n f_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ be an elliptic operator of second order. Then **its symbol is a positive definite or negative definite scalar product on T^*M.** For second order elliptic operators, **we always assume that the symbol $\sum_{i,j} f_{ij} dx_i dx_j$ is positive definite.**

REMARK: For any elliptic operator D, its symbol $\sigma(D) \in \mathrm{Sym}^2(TM)$ **defines a Riemannian metric on M.**

Strong maximum principle

THEOREM:
(strong maximum principle for second order elliptic equations; Eberhard Hopf, 1927) Let M be a manifold, not necessarily compact, and $D: C^\infty M \longrightarrow C^\infty M$ an elliptic operator of second order, which satisfies $D(const) = 0$. Consider a function $u \in C^\infty M$ such that $D(u) \geqslant 0$. **Assume that u has a local maximum somewhere on M. Then u is a constant.**

...Hopf's great paper on the maximum principle... has the beauty and elegance of a Mozart symphony, the light of a Vermeer painting. Only a fraction more than five pages in length, it contains seminal ideas which are still fresh after 75 years. – James Serrin, 2002

We start with a special case.
Proof of maximum principle, for the case $D(u) > 0$: In coordinates, D is written as $Du = \sum_{i,j} A^{ij} u_{ij} + \sum_i B^i u_i,$, where u is the matrix of second derivatives of u, $u_i = \frac{\partial u}{\partial x_i}$, and A^{ij} a function taking values in positive definite matrices. Let z be the point where u reaches a relative maximum. In this point the first derivatives of u vanish, and the matrix of second derivatives is negative semi-definite, hence $Du|_z = \sum_{i,j} A^{ij} u_{ij}|_z \leqslant 0$, contradicting $Du > 0$.
■

I will first prove **the weak maximum principle,** and then deduce the strong maximum principle.

Eberhard Hopf (1902-1983)

For a biography of E. Hopf, see "Eberhard Hopf between Germany and the US", `https://pure.mpg.de/rest/items/item_3010413_2/component/file_3015364/content`.

Weak maximum principle

THEOREM: (The weak maximum principle)
Let $D: C^\infty\mathbb{R}^n \longrightarrow C^\infty\mathbb{R}^n$ be an elliptic operator of second order, which satisfies $D(const) = 0$. Consider a relatively compact open subset $\Omega \Subset \mathbb{R}^n$. **Then any solution u of the inequality $D(u) \geqslant 0$ reaches its maximum $\sup_\Omega u$ on the boundary $\partial\Omega$.**

Proof. Step 1: Let $z \in \overline{\Omega}$ be a point where u reaches maximum, and x_i coordinates in its neighbourhood U, with origin in z. Rescaling the coordinates if necessary, we can always assume that Ω is relatively compact in the unit ball B_1. **It would suffice to show that $u(z) = u(v)$ for some $z \in \partial\Omega$.**

Step 2: Adding to u a solution φ of the inequality $D\varphi > 0$, **we obtain a function $u + \varphi$ which reaches its maximum on ∂U,** because of the strong maximum principle for solutions of $Du > 0$.

Step 3: The function $\varphi := \varepsilon e^{cx_1}$ satisfies $D\varphi > 0$ if c is chosen such that $A^{1,1}c > |B^1|$. Indeed, $\varepsilon^{-1}D(\varphi) = c^2A^{1,1}e^{cx_1} + cB^1e^{cx_1} > 0$.

Step 4: Since the maximum of $u + \varepsilon e^{cx_1}$ is reached on $\partial\Omega$ for any $\varepsilon > 0$, we obtain that $\sup_\Omega u = \lim_{\varepsilon\to 0} \sup_\Omega(u + \varepsilon e^{cx_1}) = \lim_{\varepsilon\to 0} \sup_{\partial\Omega}(u + \varepsilon e^{cx_1}) = \sup_{\partial\Omega} u$. ■

Hopf lemma

LEMMA: (Hopf lemma)
Let $D(u) = \sum_{i,j} A^{ij} u_{ij} + \sum_i B^i u_i$ be an elliptic operator on a unit ball $B \subset \mathbb{R}^n$, and $u \in C^\infty B$ a function which satisfies $D(u) \geqslant 0$. Assume that u reaches maximum $u(z_0) = 0$ in $z_0 \in \partial B$, and inside B we have $u < 0$. Denote by $\vec{r}$ the radial vector field, $\vec{r} = \sum x_i \frac{d}{dx_i}$. **Then the derivative $\operatorname{Lie}_{\vec{r}} u|_{z_0}$ in the radial direction is positive.**

Proof. Step 1: Consider a non-negative function $v \in C^\infty B$, defined by $v(x) = e^{-\alpha|x|^2} - e^{-\alpha}$, where $\alpha > 0$ is a real number. Then

$$D(v)|_x = \alpha^2 e^{-\alpha r(x)^2} \sum A^{ij} x_i x_j + e^{-\alpha r(x)^2} (\alpha\zeta + \xi),$$

where $\zeta, \xi \in \mathbb{C}^\infty B$ are bounded functions on B, independent from α. Therefore **for sufficiently big $\alpha > 0$, we have $D(v) > 0$ in the set $\Omega = B \backslash B'$,** where $B' \subset B$ is an open ball with center in 0 and radius $r_0 < 1$.

Step 2: For a sufficiently small $\varepsilon > 0$, we have $u + \varepsilon v < 0$ in B', because $u < -\delta < 0$ on B' for some $\delta > 0$. Since $v = 0$ and $u \leqslant 0$ on ∂B, weak maximum principle implies that $u + \varepsilon v < 0$ in Ω, and $u + \varepsilon v$ reaches its maximum in z_0. **This implies that $\operatorname{Lie}_{\vec{r}}(u + \varepsilon v)|_{z_0} \geqslant 0$.**

Step 3: An easy computation gives $\operatorname{Lie}_{\vec{r}} v|_{z_0} < 0$, hence Step 2 implies $\operatorname{Lie}_{\vec{r}} u|_{z_0} > 0$. ■

Strong maximum principle (proof)

THEOREM:
(strong maximum principle for second order elliptic equations; Eberhard Hopf, 1927) Let M be a manifold, not necessarily compact, and $D: C^\infty M \longrightarrow C^\infty M$ an elliptic operator of second order, which satisfies $D(const) = 0$. Consider a function $u \in C^\infty M$ such that $D(u) \geqslant 0$. **Assume that u has a local maximum somewhere on M. Then u is a constant.**

Proof. Step 1: Suppose that the local maximum is reached in an interior point $z \in M$, and $Z := \{m \in M \mid u(m) = u(z)\}$. If $u \neq const$, **there exists an open ball $B \subset M$ with interior not intersecting Z, and boundary intersecting Z.** Replacing M by an open subset, we may assume that $u(z)$ is its maximum on all M. Then B can be chosen in such a way that $u < u(z)$ inside B, and $\partial B \cap B$ is a local maximum of u.

Step 2: Since the derivative of z in z_0 is non-zero (by Hopf lemma), this point cannot be a local maximum of u, giving a contradiction. ■