Complex surfaces

lecture 8: Adjoint operators in Hodge theory

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January 24, 2024, 17:00

Adjoint connection (reminder)

DEFINITION: Given a connection ∇ on a vector bundle B equipped with a scalar product $(\cdot,\cdot)$, define ∇^* by the formula

$$d(b, b') = (\nabla(b), b') + (b, \nabla^*(b')). \quad (**)$$

Here, b, b' are sections of B, $d(b, b')$ is a differential of a function, and $(\nabla(b), b')$ is the 1-form obtained from the bilinear pairing $B \otimes (B \otimes \Lambda^1 M) \longrightarrow \Lambda^1 M$.

CLAIM: The map $\nabla^* : B \longrightarrow B \otimes \Lambda^1 M$ is well defined by (**). Moreover, it is also a connection.

Proof: The first statement is clear, because any linear map $B \longrightarrow \Lambda^1 M$ can be represented by $b \longrightarrow (b, A)$ for some $A \in B \otimes \Lambda^1 M$. To check the second statement, we take $f \in C^\infty M$, and write

$$(b, b')df + fd(b, b') = d(b, fb') = f(\nabla(b), b') + (b, \nabla^*(fb')).(**)$$

which gives $\left(b, \nabla^*(fb') - f\nabla^*(b')\right) = (b, b')df$, hence $\nabla^*(fb') - f\nabla^*(b') = b' \otimes df$.
■

DEFINITION: The connection ∇^* is called **adjoint connection** to ∇. Relation $\nabla = \nabla^*$ happens precisely when ∇ preserves the metric tensor, considered as a section of $B^* \otimes B^*$, and in this case ∇ is called **an orthogonal connection**.

Adjoint connection and L^2-product

DEFINITION: Fix a volume form Vol on a manifold M Consider a $C^\infty M$-linear scalar product on a vector bundle B. Then **the space of sections of B is also equipped with a scalar product:** $(b,b')_{L^2} = \int_M (b,b')\,\mathsf{Vol}$. It is called **the standard L^2-scalar product on the space of sections**.

LEMMA: (integration by parts)
Let B be a bundle on M with scalar product and connection ∇, and $b, b' \in B$ its sections. **Then, for any vector fields $X \in TM$, one has**

$$\int_M (\nabla_X b, b') + \int_M (b, \nabla^*_X b') = \int_M (b,b')\,\mathsf{Lie}_X \mathsf{Vol} \quad (***)$$

Proof: By definition, one has

$$\int_M ((\nabla_X)^*(b), b')\,\mathsf{Vol} = \int_M (b, \nabla_X b')\,\mathsf{Vol} = -\int_M (\nabla^*_X b, b')\,\mathsf{Vol} - \int_M \mathsf{Lie}_X (b,b')\,\mathsf{Vol},$$

where $\mathsf{Lie}_X(b,b')$ is differential of the function (b,b') along $X \in TM$. However, for any top form η, one has $\mathsf{Lie}_X(\eta) = d(i_x \eta)$ by Cartan's formula, giving $\int_M \mathsf{Lie}_X(\eta) = 0$, hence

$$0 = \int_M \mathsf{Lie}_X((b,b')\,\mathsf{Vol}) = \int_M \mathsf{Lie}_X(b,b')\,\mathsf{Vol} + \int_M (b,b')\,\mathsf{Lie}_X \mathsf{Vol},$$

giving the last term in (***). ∎

Adjoint operators

REMARK: Operators $A: F \longrightarrow G$ and $A^*: G \longrightarrow F$ on spaces with a scalar product are called **orthogonal adjoint**, or **adjoint**, if $(A(f), g) = (f, A^*(g))$ for each $f \in F$, $g \in G$.

CLAIM: An orthogonal adjoint D^* to a differential operator D **is a differential operator again.**

Proof. Step 1: This is clear for $C^\infty M$-linear operators (just take the pointwise adjoint map). **If we prove it for first order operators, we are done,** because $(XY)^* = Y^*X^*$.

Step 2: First order operators are expressed as linear combination of linear maps and derivatives $\nabla_X: F \longrightarrow F$ combined with linear maps. Therefore, **it would suffice to show that $(\nabla_X)^*$ is a differential operator.**

Step 3: The map $(\nabla_X)^*$ is a differential operator: $(\nabla_X)^*(b) = -\nabla_X^* - \frac{\operatorname{Lie}_X(\operatorname{Vol})}{\operatorname{Vol}} b$, because

$$\int_M ((\nabla_X)^*(b), b')\operatorname{Vol} = -\int_M (\nabla_X^* b, b')\operatorname{Vol} - \int (b, b')\operatorname{Lie}_X(\operatorname{Vol})$$

by "integration by parts", as shown above. ■

Laplacian on differential forms

DEFINITION: Let V be a vector space. **A metric g on V induces a natural metric on each of its tensor spaces:** $g(x_1 \otimes x_2 \otimes ... \otimes x_k, x'_1 \otimes x'_2 \otimes ... \otimes x'_k) = g(x_1, x'_1) g(x_2, x'_2) ... g(x_k, x'_k)$.

This gives a natural positive definite scalar product on differential forms over a Riemannian manifold (M, g): $g(\alpha, \beta) := \int_M g(\alpha, \beta) \operatorname{Vol}_M$.

DEFINITION: Let M be a Riemannian manifold. **Laplacian on differential forms** is $\Delta := dd^* + d^*d$.

REMARK: Laplacian is self-adjoint and positive semi-definite: $(\Delta x, x) = (dx, dx) + (d^*x, d^*x)$. Also, Δ commutes with d and d^*.

THEOREM: (The main theorem of Hodge theory)
Let Δ be the Laplacian on differential forms on a compact Riemannian manifold (or any other self-adjoint elliptic operator). Then **there is an orthonormal basis in the Hilbert space $L^2(\Lambda^*(M))$ consisting of eigenvectors of Δ,** and **its eigenvalues have finite multiplicities and converge to infinity.**

THEOREM: ("Elliptic regularity for Δ") Let $\alpha \in L^2(\Lambda^k(M))$ be an eigenvector of Δ (or any other self-adjoint elliptic operator). **Then α is a smooth k-form.**

These two theorems are foundation of Hodge theory; **throughout the course we assume them without a proof.**

Hodge theory and the cohomology

THEOREM: The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism.

Proof. Step 1: Since $d^2 = 0$ and $(d^*)^2 = 0$, one has $\{d, \Delta\} = 0$. This means that **Δ commutes with the de Rham differential.**

Step 2: Consider the eigenspace decomposition $\Lambda^*(M) \hat{=} \bigoplus_\alpha \mathcal{H}^*_\alpha(M)$, where α runs through all eigenvalues of Δ, and $\mathcal{H}^*_\alpha(M)$ is the corresponding eigenspace. **For each α, de Rham differential defines a complex**

$$\mathcal{H}^0_\alpha(M) \xrightarrow{d} \mathcal{H}^1_\alpha(M) \xrightarrow{d} \mathcal{H}^2_\alpha(M) \xrightarrow{d} \dots$$

Step 3: On $\mathcal{H}^*_\alpha(M)$, one has $dd^* + d^*d = \alpha$. When $\alpha \neq 0$, and η closed, this implies $dd^*(\eta) + d^*d(\eta) = dd^*\eta = \alpha\eta$, hence $\eta = d\xi$, with $\xi := \alpha^{-1}d^*\eta$. This implies that **the complexes $(\mathcal{H}^*_\alpha(M), d)$ don't contribute to cohomology.**

Step 4: We have proven that

$$H^*(\Lambda^* M, d) = \bigoplus_\alpha H^*(\mathcal{H}^*_\alpha(M), d) = H^*(\mathcal{H}^*_0(M), d) = \mathcal{H}^*(M).$$

■